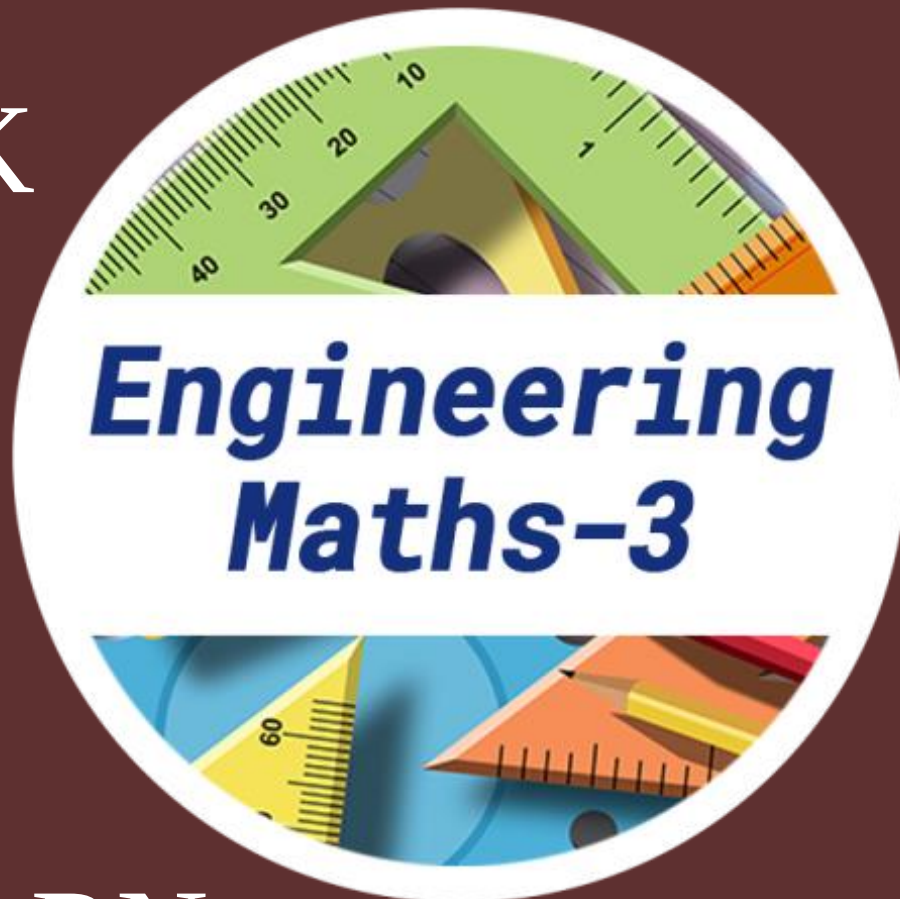


ENGINEERING MATHEMATICS

B E C O M E R E A D Y

HNK



LEARN

Mathematics 3

Mathematics is the foundation of engineering and scientific disciplines, providing essential tools and concepts for problem-solving and analysis. This textbook provides a comprehensive guide to advanced mathematical concepts, covering functions with several variables, vector calculus, differential equations, transforms, and series.

OVERVIEW

This textbook provides a comprehensive guide to advanced mathematics for engineering, focusing on essential topics such as functions with several variables, vector calculus, differential equations, and transforms. You will learn the theory and practice of these mathematical concepts, which are crucial for solving complex engineering problems.

You will learn the essentials of advanced mathematics, such as how to solve differential equations, perform vector calculus, and apply transforms like Laplace and Fourier. You will also learn how to use these mathematical tools to model and analyze engineering systems. You will apply your skills to create various projects such as analyzing dynamic systems, optimizing control processes, and solving real-world engineering problems. This textbook is suitable for anyone who wants to learn how to use advanced mathematics in a practical and innovative way.

In the second part of this textbook, you will learn advanced techniques and applications that extend the capabilities of these mathematical concepts, such as solving partial differential equations, using Z transforms for discrete systems, and applying Fourier series in signal processing.

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FISA Preparation

2.2.1 Directional derivatives, Line integral and Green's Theorem.

1. Find the directional derivative $D_{\vec{u}}f(x,y)$ if

$$f(x,y) = x^3 - 3xy + 4y^2$$

and $\vec{u}$ is the unit vector given by angle $\theta = \frac{\pi}{6}$. What is $D_{\vec{u}}f(1,2)$?

$$(1,2) \rightarrow x = 1, \quad y = 2$$

$$f(x,y) = x^3 - 3xy + 4y^2$$

$$f_x = 3x^2 - 3y = 3(1)^2 - 3(2) = -3$$

$$f_y = -3x + 8y = -3(1) + 8(2) = 13$$

$$\nabla f(x,y) = f_x \vec{i} + f_y \vec{j} = [3x^2 - 3y] \vec{i} + [-3x + 8y] \vec{j}$$

$$\nabla f(1,2) = [-3] \vec{i} + [13] \vec{j} = -3\vec{i} + 13\vec{j}$$

$$\vec{u} = \cos \theta \vec{i} + \sin \theta \vec{j} = \cos \frac{\pi}{6} \vec{i} + \sin \frac{\pi}{6} \vec{j} = \frac{\sqrt{3}}{2} \vec{i} + \frac{1}{2} \vec{j}$$

$$D_{\vec{u}}f(x,y) = \nabla f(x,y) \cdot \vec{u}$$

$$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0 \quad \vec{i} \times \vec{j} = \vec{j} \times \vec{k} = \vec{k} \times \vec{i} = \vec{i} \times \vec{k} = \vec{j} \times \vec{i} = \vec{k} \times \vec{j} = -\vec{i}$$

$$D_{\vec{u}}f(x,y) = [f_x \vec{i} + f_y \vec{j}] \cdot [\cos \theta \vec{i} + \sin \theta \vec{j}] = [f_x \cos \theta + f_y \sin \theta]$$

$$D_{\vec{u}}f(x,y) = [-3\vec{i} + 13\vec{j}] \cdot \left[\frac{\sqrt{3}}{2} \vec{i} + \frac{1}{2} \vec{j} \right]$$

$$D_{\vec{u}}f(x,y) = \left[-3 \times \frac{\sqrt{3}}{2} \right] + \left[13 \times \frac{1}{2} \right] = -\frac{3\sqrt{3}}{2} + \frac{13}{2} = \frac{13 - 3\sqrt{3}}{2} = 3.902$$

2. Find the directional derivative of the function

$$f(x,y) = x^2 y^3 - 4y$$

at the point $(2,-1)$ in the direction of the vector $\vec{v} = 2\vec{i} + 5\vec{j}$.

$$(2,-1) \rightarrow x = 2, \quad y = -1$$

$$f(x,y) = x^2 y^3 - 4y$$

$$f_x = 2xy^3 = 2(2)(-1)^3 = -6$$

$$f_y = 3x^2 y^2 - 4 = 3(2)^2 (-1)^2 - 4 = 8$$

$$\nabla f(x,y) = f_x \vec{i} + f_y \vec{j} = [2xy^3] \vec{i} + [3x^2 y^2 - 4] \vec{j}$$

$$\nabla f(2,-1) = [-6] \vec{i} + [8] \vec{j} = -6\vec{i} + 8\vec{j}$$

$$\vec{u} = \cos \theta \vec{i} + \sin \theta \vec{j}$$

$$\vec{v} = 2\vec{i} + 5\vec{j}$$

$$|\vec{v}| = \sqrt{(2)^2 + (5)^2} = \sqrt{29}$$

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{2\vec{i} + 5\vec{j}}{\sqrt{29}} = \frac{2}{\sqrt{29}} \vec{i} + \frac{5}{\sqrt{29}} \vec{j}$$

$$D_{\vec{u}}f(x,y) = \nabla f(x,y) \cdot \vec{u}$$

$$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0 \quad \vec{i} \times \vec{j} = \vec{j} \times \vec{k} = \vec{k} \times \vec{i} = \vec{i} \times \vec{k} = \vec{j} \times \vec{i} = \vec{k} \times \vec{j} = -\vec{i}$$

$$D_{\vec{u}}f(x,y) = [f_x \vec{i} + f_y \vec{j}] \cdot [\vec{u}]$$

$$\begin{aligned} \text{Duf}(x,y) &= [-6i + 8j] \times \left[\frac{2}{\sqrt{29}}i + \frac{5}{\sqrt{29}}j \right] \\ \text{Duf}(x,y) &= \left[-6 \times \frac{2}{\sqrt{29}} \right] + \left[8 \times \frac{5}{\sqrt{29}} \right] = -\frac{12}{\sqrt{29}} + \frac{40}{\sqrt{29}} = \frac{28}{\sqrt{29}} = \frac{28\sqrt{29}}{29} = 5.199469469 \end{aligned}$$

3. Find the directional derivative of
 $f(x,y) = \sqrt{xy}$
 at P(2, 8) in the direction of Q(5, 4).

$$\begin{aligned} f(x,y) &= \sqrt{xy} = (xy)^{\frac{1}{2}} = x^{\frac{1}{2}}y^{\frac{1}{2}} = \sqrt{x}\sqrt{y} \\ f_x &= [\sqrt{xy}]' = \left[(xy)^{\frac{1}{2}} \right]' = \left[x^{\frac{1}{2}}y^{\frac{1}{2}} \right]' = y^{\frac{1}{2}} \left[x^{\frac{1}{2}} \right]' = \sqrt{y} \left[x^{\frac{1}{2}} \right]' = \frac{\sqrt{y}}{2} (x)^{\frac{1}{2}-1} = \frac{\sqrt{y}}{2} (x)^{-\frac{1}{2}} \\ &= \frac{\sqrt{y}}{2} (x)^{-\frac{1}{2}} = \frac{\sqrt{y}}{2(x)^{\frac{1}{2}}} = \frac{\sqrt{y}}{2\sqrt{x}} = \frac{1}{2} \sqrt{\frac{y}{x}} \\ f_y &= [\sqrt{xy}]' = \left[(xy)^{\frac{1}{2}} \right]' = \left[x^{\frac{1}{2}}y^{\frac{1}{2}} \right]' = x^{\frac{1}{2}} \left[y^{\frac{1}{2}} \right]' = \sqrt{x} \left[y^{\frac{1}{2}} \right]' = \frac{\sqrt{x}}{2} (y)^{\frac{1}{2}-1} = \frac{\sqrt{x}}{2} (y)^{-\frac{1}{2}} = \frac{\sqrt{x}}{2(y)^{\frac{1}{2}}} \\ &= \frac{\sqrt{x}}{2\sqrt{y}} \\ &= \frac{1}{2} \sqrt{\frac{x}{y}} \end{aligned}$$

$$\begin{aligned} f_x &= \frac{\sqrt{y}}{2\sqrt{x}} = \frac{1}{2} \sqrt{\frac{y}{x}} \\ f_y &= \frac{\sqrt{x}}{2\sqrt{y}} = \frac{1}{2} \sqrt{\frac{x}{y}} \end{aligned}$$

$$\begin{aligned} &\text{P(2, 8) in the direction of Q(5, 4)} \\ Q(5, 4) &\rightarrow \vec{Q}(5, 4) = (5i + 4j) \\ P(2, 8) &\rightarrow \vec{P}(2, 8) = (2i + 8j) \\ \overrightarrow{PQ} &= \vec{Q} - \vec{P} = (5i + 4j) - (2i + 8j) = 5i + 4j - 2i - 8j = (5i - 2i) + (4j - 8j) \\ &= (5 - 2)i + (4 - 8)j = 3i - 4j \\ \overrightarrow{PQ} &= 3i - 4j \\ |\overrightarrow{PQ}| &= \sqrt{a^2 + b^2} = \sqrt{(3)^2 + (4)^2} = 5 \\ \vec{u} &= \frac{3i - 4j}{5} = \frac{3i}{5} - \frac{4j}{5} = \frac{3}{5}i - \frac{4}{5}j \end{aligned}$$

$$\begin{aligned} \text{Duf}(x,y) &= \nabla(x,y) \times \vec{u} \\ i \times i &= j \times j = k \times k = 1 \quad i \times j = i \times k = j \times k = 0 \end{aligned}$$

$$\begin{aligned}
 \text{Duf}(x, y) &= [f_x i + f_y j] \times [\vec{u}] \\
 \text{Duf}(x, y) &= \left[\frac{\sqrt{y}}{2\sqrt{x}} + \frac{\sqrt{x}}{2\sqrt{y}} \right] \times \left[\frac{3}{5} i - \frac{4}{5} j \right] \\
 \text{Duf}(x, y) &= \left[\frac{\sqrt{y}}{2\sqrt{x}} \times \frac{3}{5} \right] + \left[\frac{\sqrt{x}}{2\sqrt{y}} \times -\frac{4}{5} \right] = \left[\frac{3\sqrt{y}}{10\sqrt{x}} \right] - \left[\frac{2\sqrt{x}}{5\sqrt{y}} \right] = \left[\frac{3}{10} \right] \frac{\sqrt{y}}{\sqrt{x}} - \left[\frac{2}{5} \right] \frac{\sqrt{x}}{\sqrt{y}} \\
 &= \left[\frac{3}{10} \right] \sqrt{\frac{y}{x}} - \left[\frac{2}{5} \right] \sqrt{\frac{x}{y}} = \frac{3}{10} \sqrt{\frac{y}{x}} - \frac{2}{5} \sqrt{\frac{x}{y}} = 0.3 \sqrt{\frac{y}{x}} - 0.4 \sqrt{\frac{x}{y}}
 \end{aligned}$$

$$\text{Duf}(x, y) = \frac{3}{10} \sqrt{\frac{y}{x}} - \frac{2}{5} \sqrt{\frac{x}{y}} = 0.3 \sqrt{\frac{y}{x}} - 0.4 \sqrt{\frac{x}{y}}$$

$$\begin{aligned}
 P(2, 8) &\rightarrow \rightarrow x = 2, \quad y = 8 \\
 \text{Duf}(2, 8) &= \frac{3}{10} \sqrt{\frac{8}{2}} - \frac{2}{5} \sqrt{\frac{2}{8}} = 0.3 \sqrt{\frac{8}{2}} - 0.4 \sqrt{\frac{2}{8}} = \frac{2}{5} = 0.4
 \end{aligned}$$

$$\text{Duf}(2, 8) = \frac{2}{5} = 0.4$$

4. Find the directional derivative of

$$f(x, y, z) = xy^2z^3$$

at P(2, 1, 1) in the direction of Q(0, -3, 5)

$$P(2, 1, 1) \rightarrow x = 2, \quad y = 1, \quad z = 1$$

$$f(x, y, z) = xy^2z^3$$

$$f_x = y^2z^3 = (1)(1)^3 = 1$$

$$f_y = 2xyz^3 = 2(2)(1)(1)^3 = 4$$

$$f_z = 3xy^2z^2 = 3(2)(1)^2(1)^3 = 6$$

$$\nabla(x, y, z) = f_x i + f_y j + f_z k$$

$$\nabla(2, 1, 1) = [1]i + [4]j + [6]k = 1i + 4j + 6k$$

P(2, 1, 1) in the direction of Q(0, -3, 5)

$$P(2, 1, 1) \rightarrow \vec{P}(2, 1, 1) = (2i + 1j + 1k)$$

$$Q(0, -3, 5) \rightarrow \vec{Q}(0, -3, 5) = (0i - 3j + 5k)$$

$$\begin{aligned}
 \overrightarrow{PQ} &= \vec{Q} - \vec{P} = (0i - 3j + 5k) - (2i + 1j + 1k) = 0i - 3j + 5k - 2i - 1j - 1k \\
 &= (0i - 2i) + (-3j - 1j) + (5k - 1k) = (0 - 2)i + (-3 - 1)j + (5 - 1)k \\
 &= (-2)i + (-4)j + (4)k = -2i - 4j + 4k
 \end{aligned}$$

$$\overrightarrow{PQ} = -2i - 4j + 4k$$

$$\begin{aligned}
 |\overrightarrow{PQ}| &= \sqrt{a^2 + b^2} = \sqrt{(-2)^2 + (-4)^2 + (4)^2} = 6 \\
 \vec{u} &= \frac{-2i - 4j + 4k}{6} = -\frac{2i}{6} - \frac{4j}{6} + \frac{4k}{6} = -\frac{1}{3}i - \frac{2}{3}j + \frac{2}{3}k
 \end{aligned}$$

$$\begin{aligned}
& \text{Duf}(x, y, z) = \nabla(x, y, z) \times \vec{u} \\
& i \times i = j \times j = k \times k = 1 \quad i \times j = i \times k = j \times k = 0 \\
& \text{Duf}(x, y, z) = [f_x i + f_y j + f_z k] \times [\vec{u}] \\
& \text{Duf}(x, y, z) = [1i + 4j + 6k] \times \left[-\frac{1}{3}i - \frac{2}{3}j + \frac{2}{3}k\right] \\
& \text{Duf}(x, y, z) = \left[1 \times -\frac{1}{3}\right] + \left[4 \times -\frac{2}{3}\right] + \left[6 \times \frac{2}{3}\right] = -\left[\frac{1}{3}\right] - \left[\frac{8}{3}\right] + \left[\frac{12}{3}\right] = -\frac{1}{3} - \frac{8}{3} + 4 = 1 \\
& \text{Duf}(x, y, z) = 1
\end{aligned}$$

5. Find the directions in which the directional derivative of
 $f(x, y) = x^2 + xy^3$
at the point (2, 1) has the value 2.

$$\begin{aligned}
& \text{Duf}(x, y) = 2 \\
& (2, 1) \rightarrow x = 2, \quad y = 1 \\
& f(x, y) = x^2 + xy^3 \\
& f_x = 2x + y^3 = 2(2) + (1)^3 = 5 \\
& f_y = 3xy^2 = 3(2)(1)^2 = 6 \\
& \nabla(x, y) = f_x i + f_y j = [2x + y^3]i + [3xy^2]j \\
& \nabla(2, 1) = [5]i + [6]j = 5i + 6j \\
& \vec{u} = i \cos \theta + j \sin \theta = ? \\
& \text{Duf}(x, y) = \nabla(x, y) \times \vec{u} \\
& i \times i = j \times j = k \times k = 1 \quad i \times j = i \times k = j \times k = 0 \\
& \text{Duf}(x, y) = [f_x i + f_y j] \times [i \cos \theta + j \sin \theta] = [f_x \cos \theta + f_y \sin \theta] \\
& \text{Duf}(x, y) = 2 \\
& \text{Duf}(x, y) = \nabla(x, y) \times \vec{u} \\
& 2 = \nabla(2, 1) \times \vec{u} \\
& 2 = [5i + 6j] \times \vec{u} \\
& \frac{2}{[5i + 6j]} = \frac{[5i + 6j]}{[5i + 6j]} \times \vec{u} \\
& \vec{u} = \frac{2}{[5i + 6j]} = \frac{2}{[5i + 6j]} \frac{[5i - 6j]}{[5i - 6j]} \\
& = \frac{2[5i - 6j]}{2[5i - 6j]} = \frac{2[5i - 6j]}{[(5i \times 5i) - (5i \times 6j) + (5i \times 6j) - (6j \times 6j)]} = \frac{2[5i - 6j]}{(5i \times 5i) - (6j \times 6j)} \\
& = \frac{2[5i - 6j]}{(5i)^2 - (6j)^2} = \frac{2[5i - 6j]}{-(5)^2 + (6)^2} = \frac{2[5i - 6j]}{(6)^2 - (5)^2} = \frac{2[5i - 6j]}{36 - 25} = \frac{2[5i - 6j]}{11} \\
& \vec{u} = \frac{10i - 12j}{11} = \frac{10}{11}i - \frac{12}{11}j \\
& |u| = \sqrt{a^2 + b^2} = \sqrt{\left(\frac{10}{11}\right)^2 + \left(\frac{12}{11}\right)^2} = \frac{2\sqrt{61}}{11} = 1.420045396
\end{aligned}$$

$$\theta = \tan^{-1} \left[-\frac{\frac{12}{11}}{\frac{10}{11}} \right] = \tan^{-1} \left[-\frac{12}{11} \frac{11}{10} \right] = \tan^{-1} \left[-\frac{12}{10} \right] = \tan^{-1}(-1.2) = -0.8760580506$$

$$\vec{u} = \frac{10}{11}i - \frac{12}{11}j = \frac{2\sqrt{61}}{11}e^{-j0.876} = 1.42e^{-j0.876}$$

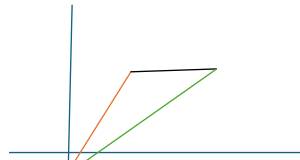
x axis & y axis

$$\theta = -0.8760580506$$

6. Use line integral for the contour integral

$$\oint (3x^2 + xy)dx + (6x^3 + 3y^2)dy$$

where C is a triangle with vertices (0, 0), (2, 2), (1, 2).



$A(0,0), B(2,2), C(1,2)$

$$A(0,0) = (x_1, y_1) \rightarrow x_1 = 0, \quad y_1 = 0$$

$$B(2,2) = (x_2, y_2) \rightarrow x_2 = 2, \quad y_2 = 2$$

$$C(1,2) = (x_3, y_3) \rightarrow x_3 = 1, \quad y_3 = 2$$

$(A \rightarrow B)$ vertices $(0,0)(2,2)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 0}{2 - 0} = 1$$

tangent line

$$y - y_1 = m(x - x_1) \quad \text{Or} \quad y - y_2 = m(x - x_2)$$

$$y - 0 = 1(x - 0) \quad y - 2 = 1(x - 2)$$

$$y - 0 = x - 0 \quad y - 2 = x - 2$$

$$y = x \quad y = x - 2 + 2 = x$$

$(A \rightarrow C)$ vertices $(0,0)(1,2)$

$$m = \frac{y_3 - y_1}{x_3 - x_1} = \frac{2 - 0}{1 - 0} = 2$$

tangent line

$$y - y_1 = m(x - x_1) \quad \text{Or} \quad y - y_3 = m(x - x_3)$$

$$y - 0 = 2(x - 0) \quad y - 2 = 2(x - 1)$$

$$y - 0 = 2x - 0 \quad y - 2 = 2x - 2$$

$$y = 2x \quad y = 2x - 2 + 2 = 2x$$

$(B \rightarrow C)$ vertices $(2,2)(1,2)$

$$m = \frac{y_3 - y_2}{x_3 - x_2} = \frac{2 - 2}{1 - 2} = 0$$

tangent line

$$\begin{array}{ll} y - y_2 = m(x - x_2) & \text{Or} \quad y - y_3 = m(x - x_3) \\ y - 2 = 0(x - 2) & y - 2 = 0(x - 1) \\ y - 2 = 0 & y - 2 = 0 \\ y = 2 & y = 2 \end{array}$$

now, let's start with the work

$$\oint (3x^2 + xy)dx + (6x^3 + 3y^2)dy$$

(A → B) vertices (0, 0)(2, 2)

$$y = x$$

$$dx = dy$$

$$\int (3x^2 + xy)dx + (6x^3 + 3y^2)dy$$

$$\int (3x^2 + xx)dx + (6x^3 + 3x^2)dx \quad \text{Or} \quad \int (3y^2 + yy)dy + (6y^3 + 3y^2)dy$$

$$\int (3x^2 + x^2)dx + (6x^3 + 3x^2)dx \quad \int (3y^2 + y^2)dy + (6y^3 + 3y^2)dy$$

$$\int (4x^2)dx + (6x^3 + 3x^2)dx \quad \int (4y^2)dy + (6y^3 + 3y^2)dy$$

$$\int (4x^2 + 6x^3 + 3x^2)dx \quad \int (4y^2 + 6y^3 + 3y^2)dy$$

$$\int (7x^2 + 6x^3)dx \quad \int (7y^2 + 6y^3)dy$$

let's try with the line between (A → C)

$$\oint (3x^2 + xy)dx + (6x^3 + 3y^2)dy$$

(C → A) vertices (1, 2)(0, 0)

$$y = 2x \quad x = \frac{1}{2}y$$

derivative both side

$$dy = 2dx \quad dx = \frac{1}{2}dy$$

$$\int (3x^2 + xy)dx + (6x^3 + 3y^2)dy$$

$$\int (3x^2 + xy)dx + (6x^3 + 3y^2)dy \quad \text{Or} \quad \int (3x^2 + xy)dx + (6x^3 + 3y^2)dy$$

$$\int (3x^2 + 2xx)dx + 2[6x^3$$

$$+ 3(2x)^2]dx$$

$$\int \left[3\left(\frac{1}{2}y\right)^2 + \left(\frac{1}{2}y\right)y \right] \frac{1}{2}dy + \left[6\left(\frac{1}{2}y\right)^3 + 3y^2 \right] dy$$

$$\int (3x^2 + 2x^2)dx + 2[6x^3 + 6x^2]dx + 3y^2] dy$$

$$\int (5x^2)dx + [12x^3 + 12x^2]dx$$

$$\int [5x^2 + 12x^3 + 12x^2] dx$$

$$\int [12x^3 + 17x^2] dx$$

$$\int \left[\frac{3}{4}y^2 + \frac{1}{2}y^2 \right] \frac{1}{2} dy + \left[\frac{6}{8}y^3 \right]$$

$$\int \left[\frac{5}{8} \right] y^2 dy + \left[\frac{6}{8}y^3 + 3y^2 \right] dy$$

$$\int \left[\frac{5}{8}y^2 + \frac{6}{8}y^3 + 3y^2 \right] dy$$

$$\int \left[\frac{29}{8}y^2 + \frac{6}{8}y^3 \right] dy$$

the last one

(B → C) vertices (2,2)(1, 2)

$$y = 2$$

$$dy = 0 dx$$

$$\int (3x^2 + xy)dx + (6x^3 + 3y^2)dy$$

$$\int (3x^2 + 2x)dx + 0(6x^3 + 3x^2)dx$$

$$\int (3x^2 + 2x) dx$$

bounde

$$\oint = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \nabla dx dy$$

$$\int_2^1 (3x^2 + 2x)dx$$

$dx \rightarrow (B, A)$

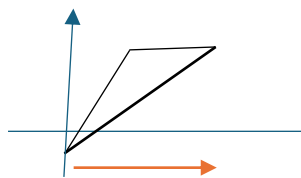
midfielder

$$(1 < x < 2)$$



endpoint
 $dy \rightarrow (C, A)$
 $(2 > y > 0)$

$$\int_2^0 \left(\frac{29}{8}y^2 + \frac{6}{8}y^3 \right) dy$$



$dx \rightarrow (A, B)$

starting point

$$(0 < x < 2)$$

$$\int_0^2 (7x^2 + 6x^3)dx$$

$$\oint = \int_0^2 (7x^2 + 6x^3) dx + \int_2^0 \left(\frac{29}{8} y^2 + \frac{6}{8} y^3 \right) dy + \int_2^1 (3x^2 + 2x) dx$$

$$A(0,0), B(2,2), C(1,2)$$

$$A(0,0) = (x_1, y_1) \rightarrow x_1 = 0, \quad y_1 = 0$$

$$B(2,2) = (x_2, y_2) \rightarrow x_2 = 2, \quad y_2 = 2$$

$$C(1,2) = (x_3, y_3) \rightarrow x_3 = 1, \quad y_3 = 2$$

$$(A \rightarrow B) \text{ vertices } (0,0)(2,2)$$

$$x_1 = 0, \quad y_1 = 0$$

$$x_2 = 2, \quad y_2 = 2$$

$$\text{contour integral}$$

$$\oint = \int_{x_1}^{x_2} \nabla dx = \int_0^2 \nabla dx$$

$$\int (7x^2 + 6x^3) dx$$

$$\int_0^2 (7x^2 + 6x^3) dx$$

$$\int_0^2 7x^2 dx + \int_0^2 6x^3 dx = \frac{128}{3}$$

$$(C \rightarrow A) \text{ vertices } (1,2)(0,0)$$

$$x_1 = 0, \quad y_1 = 0$$

$$x_3 = 1, \quad y_3 = 2$$

$$\text{contour integral}$$

$$\oint = \int_{y_3}^{y_1} \nabla dy = \int_2^0 \nabla dy$$

$$\int \left(\frac{29}{8} y^2 + \frac{6}{8} y^3 \right) dy$$

$$\int_2^0 \left(\frac{29}{8} y^2 + \frac{6}{8} y^3 \right) dy$$

$$\int_2^0 \frac{29}{8} y^2 dy + \int_2^0 \frac{6}{8} y^3 dy = -\frac{38}{3}$$

$$(B \rightarrow C) \text{ vertices } (2,2)(1,2)$$

$$x_1 = 1 \quad x_3 = 2$$

$$\text{contour integral}$$

$$\oint = \int_{x_3}^{x_1} \nabla dx = \int_2^1 \nabla dx$$

$$\int (3x^2 + 2x) dx$$

$$\int_2^1 (3x^2 + 2x) dx$$

$$\int_2^1 3x^2 dx + \int_2^1 2x dx = -10$$

Last step:

$$\oint = \int_0^2 (7x^2 + 6x^3) dx + \int_2^0 \left(\frac{29}{8} y^2 + \frac{6}{8} y^3 \right) dy + \int_2^1 (3x^2 + 2x) dx$$

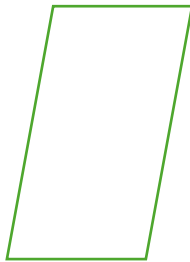
$$\oint = \frac{128}{3} - \frac{38}{3} - 10 = 20$$

$$\oint = 20$$

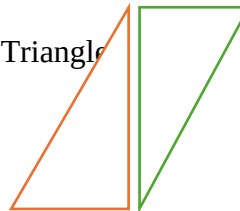
7. If the region C is represented by the parallelogram with vertices (0, 0), (1, 0), (3, 2), (2, 2), use the line integral to evaluate

$$\oint = (4x^2 + y^2) dx + (2x^2 - y^2) dy$$

A(0, 0), B(1, 0), C(3, 2), D(2, 2)



Parallelogram = 2 Triangle



A(0, 0), B(1, 0), C(3, 2), D(2, 2)

$$\begin{aligned} A(0, 0) &= (x_1, y_1) \rightarrow x_1 = 0, & y_1 &= 0 \\ B(1, 0) &= (x_2, y_2) \rightarrow x_2 = 1, & y_2 &= 0 \\ C(3, 2) &= (x_3, y_3) \rightarrow x_3 = 3, & y_3 &= 2 \\ D(2, 2) &= (x_3, y_3) \rightarrow x_3 = 2, & y_3 &= 2 \end{aligned}$$

(A → B) vertices (0, 0)(1, 0)

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 0}{1 - 0} = 0$$

tangent line

$$\begin{aligned} y - y_1 &= m(x - x_1) & \text{Or} & & y - y_2 &= m(x - x_2) \\ y - 0 &= 0(x - 0) & & & y - 0 &= 0(x - 1) \\ y - 0 &= 0(x) & & & y - 0 &= 0 \\ \mathbf{y} &= \mathbf{0} & & & \mathbf{y} &= \mathbf{0} \end{aligned}$$

(B → C) vertices (1, 0)(3, 2)

$$m = \frac{y_3 - y_2}{x_3 - x_2} = \frac{2 - 0}{3 - 1} = -1$$

tangent line

$$\begin{aligned} y - y_2 &= m(x - x_2) & \text{Or} & & y - y_3 &= m(x - x_3) \\ y - 0 &= -1(x - 1) & & & y - 2 &= -1(x - 3) \\ y - 0 &= 1 - x & & & y - 2 &= 3 - x \\ \mathbf{y} &= \mathbf{1 - x} & & & \mathbf{y} &= \mathbf{1 - x} \end{aligned}$$

(C → D) vertices (3, 2)(2, 2)

$$m = \frac{y_4 - y_3}{x_4 - x_3} = \frac{2 - 2}{2 - 3} = 0$$

tangent line

$$\begin{array}{ll} y - y_3 = m(x - x_3) & \text{Or} \quad y - y_4 = m(x - x_4) \\ y - 2 = 0(x - 3) & y - 2 = 0(x - 2) \\ y - 2 = 0 & y - 2 = 0 \\ \mathbf{y = 2} & \mathbf{y = 2} \end{array}$$

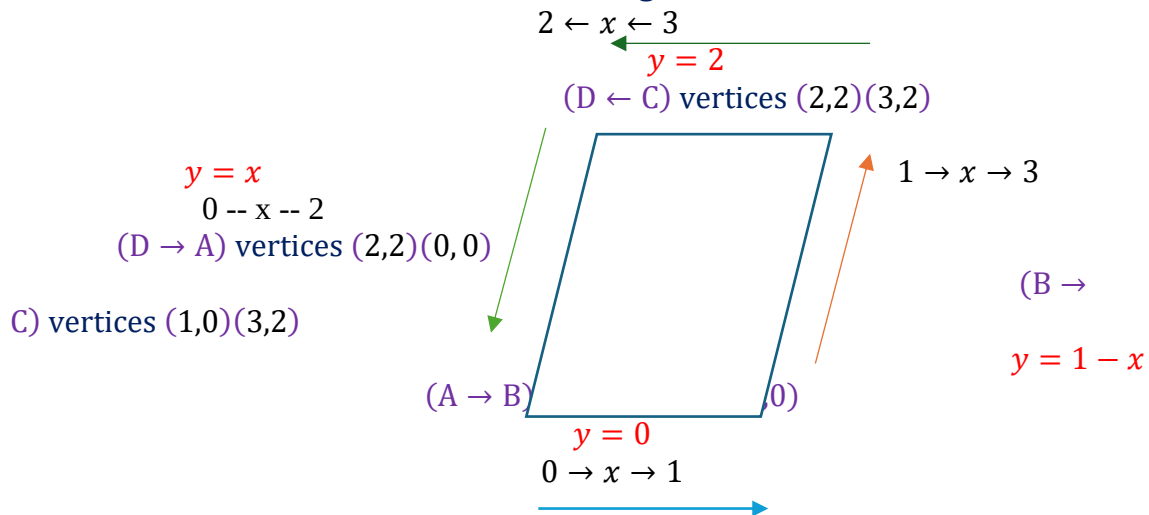
(D → A) vertices (2,2)(0,0)

$$m = \frac{y_4 - y_1}{x_4 - x_1} = \frac{0 - 2}{0 - 2} = 1$$

tangent line

$$\begin{array}{ll} y - y_4 = m(x - x_4) & \text{Or} \quad y - y_1 = m(x - x_1) \\ y - 2 = 1(x - 2) & y - 0 = 1(x - 0) \\ y - 2 = x - 2 & y - 0 = x \\ \mathbf{y = x} & \mathbf{y = x} \end{array}$$

Parallelogram:



(0,0)(1,0)

$$\mathbf{y = 0}$$

$$dy = 0dx$$

$$\oint (4x^2 + y^2)dx + (2x^2 - y^2)dy \quad \text{intervals} \quad 0 \rightarrow x \rightarrow 1$$

$$\int_0^1 (4x^2) dx = \int_0^1 4x^2 dx = \frac{4}{3}$$

(2,2)(3,2)

$$\mathbf{y = 2}$$

$$dy = 0dx$$

$$\oint (4x^2 + y^2)dx + (2x^2 - y^2)dy \quad \text{intervals} \quad 2 \leftarrow x \leftarrow 3$$

$$\int_3^2 (4x^2 + y^2)dx + (2x^2 - y^2)dy = \int_2^3 (4x^2 + 2^2)dx = -\frac{88}{3}$$

(2,2)(0,0)

$$y = x$$

$$dy = dx$$

$$\oint (4x^2 + y^2)dx + (2x^2 - y^2)dy \quad \text{intervals} \quad 2 \rightarrow x \rightarrow 0$$

$$\int_2^0 (4x^2 + x^2)dx + (2x^2 - x^2)dx = \int_2^0 6x^2 dx = -16$$

(1,0)(3,2)

$$y = 1 - x$$

$$dy = -dx$$

$$\oint (4x^2 + y^2)dx + (2x^2 - y^2)dy \quad \text{intervals} \quad 1 \rightarrow x \rightarrow 3$$

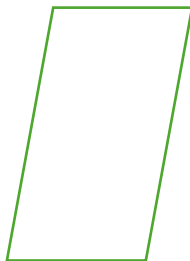
$$\int_1^3 (4x^2 + (1-x)^2)dx - (2x^2 - (1-x)^2)dx = \int_1^3 6x^2 dx = 52$$

$$\oint = -\frac{88}{3} - 16 + \frac{4}{3} + 52 = 8$$

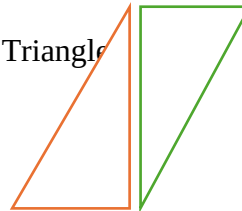
8. Apply Green's theorem to evaluate

$$\oint (4x^2 + y^2)dx + (2x^2 - y^2)dy$$

dy where C is represented by the parallelogram with vertices (0,0), (1,1), (0,1), (-1,0)



Parallelogram = 2 Triangle



A(0,0), B(1,1), C(0,1), D(-1,0)

$$A(0,0) = (x_1, y_1) \rightarrow x_1 = 0, \quad y_1 = 0$$

$$B(1,1) = (x_2, y_2) \rightarrow x_2 = 1, \quad y_2 = 1$$

$$C(0,1) = (x_3, y_3) \rightarrow x_3 = 0, \quad y_3 = 1$$

$$D(-1,0) = (x_4, y_4) \rightarrow x_4 = -1, \quad y_4 = 0$$

(A → B) vertices (0,0)(1,1)

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 0}{1 - 0} = 1$$

tangent line

$$\begin{array}{ll} y - y_1 = m(x - x_1) & \text{Or} \quad y - y_2 = m(x - x_2) \\ y - 0 = 1(x - 0) & y - 1 = 1(x - 1) \\ y - 0 = x & y - 1 = x - 1 \\ \mathbf{y = x} & \mathbf{y = x} \end{array}$$

(B → C) vertices (1, 1)(0, 1)

$$m = \frac{y_3 - y_2}{x_3 - x_2} = \frac{1 - 1}{0 - 1} = 0$$

tangent line

$$\begin{array}{ll} y - y_2 = m(x - x_2) & \text{Or} \quad y - y_3 = m(x - x_3) \\ y - 1 = 0(x - 1) & y - 1 = 0(x - 0) \\ y - 1 = 0 & y - 1 = 0 \\ \mathbf{y = 1} & \mathbf{y = 1} \end{array}$$

(C → D) vertices (0, 1)(-1, 0)

$$m = \frac{y_4 - y_3}{x_4 - x_3} = \frac{0 - 1}{-1 - 0} = 1$$

tangent line

$$\begin{array}{ll} y - y_3 = m(x - x_3) & \text{Or} \quad y - y_4 = m(x - x_4) \\ y - 1 = 1(x - 0) & y - 0 = 1(x + 1) \\ y - 1 = x & y - 0 = x + 1 \\ \mathbf{y = 1 + x} & \mathbf{y = x + 1} \end{array}$$

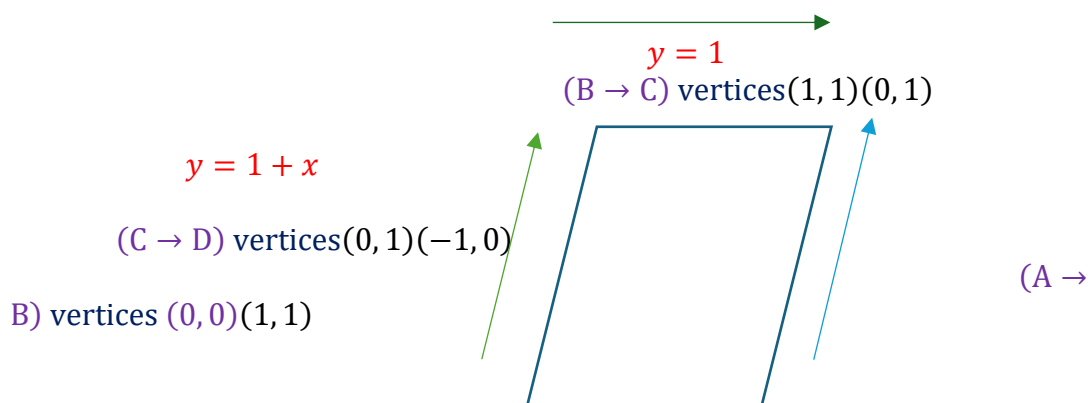
(D → A) vertices (-1, 0)(0, 0)

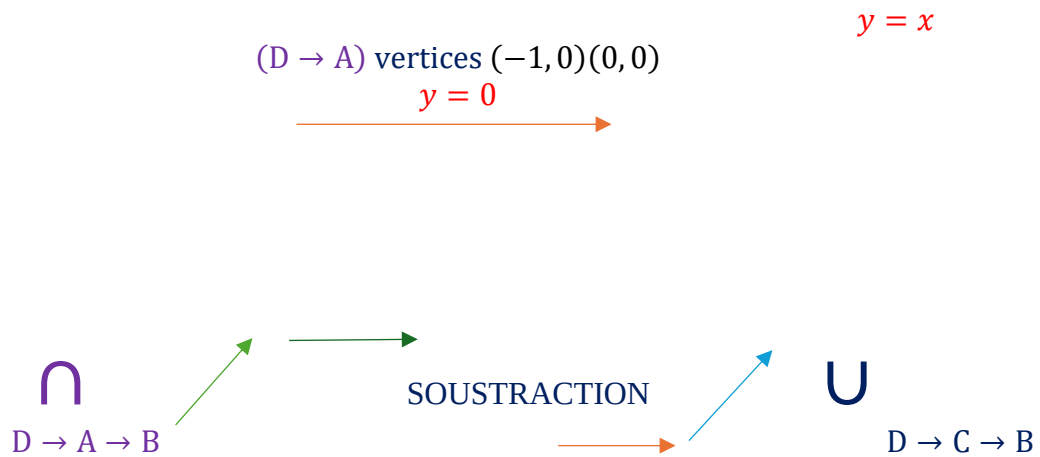
$$m = \frac{y_1 - y_4}{x_1 - x_4} = \frac{0 - 0}{0 - (-1)} = 0$$

tangent line

$$\begin{array}{ll} y - y_4 = m(x - x_4) & \text{Or} \quad y - y_1 = m(x - x_1) \\ y - 0 = 0(x + 1) & y - 0 = 0(x - 0) \\ y - 0 = 0 & y - 0 = 0 \\ \mathbf{y = 0} & \mathbf{y = 0} \end{array}$$

Parallelogram:





8. Apply Green's theorem to evaluate

$$\oint (2x^2 + y^2)dx + (2x - y^2) dy$$

dy where C is represented by the parallelogram with vertices $(0, 0), (1, 1), (0, 1), (-1, 0)$

$$\oint (2x^2 + y^2)dx + (2x - y^2) dy$$

$$\nabla = \left| \begin{matrix} (2x^2 + y^2) & (2x - y^2) \\ dx & dy \end{matrix} \right| = -[2x^2 + y^2]dy + [2x - y^2]dx = -[2y] + [2] = -2y + 2$$

$$\oint = \int_{x_3}^{x_2} \int_{\cap}^{\cup} \nabla dx dy = \int_{-1}^1 \int_{\cap}^{\cup} \nabla dx dy$$

$$\int \nabla dy = \int [-2y + 2] dy = [-y^2 + 2y] = \text{⌘}$$

$$\int_{\cap}^{\cup} = \int_{-1}^0 (\text{⌘}) dx + \int_0^1 (\text{⌘}) dx - \int_{-1}^0 (\text{⌘}) dx - \int_0^1 (\text{⌘}) dx$$

$$\int_{\cap}^{\cup} = \int_{-1}^0 [-y^2 + 2y] dx + \int_0^1 [-y^2 + 2y] dx - \int_{-1}^0 [-y^2 + 2y] dx - \int_0^1 [-y^2 + 2y] dx$$

$$\int_{-1}^0 \leftarrow (D, A) \rightarrow y = 0$$

$$[-y^2 + 2y] = [-(0)^2 + 2(0)] = [0]$$

$$\int_0^1 \leftarrow (A, B) \rightarrow y = x$$

$$[-y^2 + 2y] = [-(x)^2 + 2(x)] = [-x^2 + 2x]$$

$$\int_0^1 \leftarrow (B, C) \rightarrow y = 1$$

$$[-y^2 + 2y] = [-(1)^2 + 2(1)] = [1]$$

$$\int_{-1}^0 \leftarrow (C, D) \rightarrow y = x + 1$$

$$[-y^2 + 2y] = [-(x+1)^2 + 2(x+1)] = [-(x+1)(x+1) + 2x + 2]$$

$$= [-(x^2 + 2x + 1) + 2x + 2]$$

$$= [-x^2 - 2x - 1 + 2x + 2] = [-x^2 + 1]$$

$$\int_{\cap}^{\cup} = (D, A) + (A, B) - (B, C) - (C, D)$$

green's theorem

$$\oint = \int_{x_3}^{x_1} \int_{\cap}^{\cup} \nabla dx dy = \int_{-1}^1 \int_{\cap}^{\cup} \nabla dx dy$$

integrate

$$\oint = \int_{x_3}^{x_1} \int_{\cap}^{\cup} \nabla dx dy = \int_{-1}^1 \int_{\cap}^{\cup} \nabla dx dy$$

$$\int \nabla dy = \int [2x + 4y] dy = [2xy + 2y^2] = \textcolor{red}{x}$$

$$\int_{\cap}^{\cup} = \int_{-1}^0 (\textcolor{red}{x}) dx + \int_0^1 (\textcolor{red}{x}) dx - \int_{-1}^0 (\textcolor{red}{x}) dx - \int_0^1 (\textcolor{red}{x}) dx$$

$$\int_{\cap}^{\cup} = \int_{-1}^0 [-2y + 4x] dx + \int_0^1 [-2y + 4x] dx - \int_{-1}^0 [-2y + 4x] dx - \int_0^1 [-2y + 4x] dx$$

conversion

$$\int_{\cap}^{\cup} = (D, A) + (A, B) - (B, C) - (C, D)$$

$$\int_{\cap}^{\cup} = \int_{-1}^0 [0] dx + \int_0^1 [-x^2 + 2x] dx - \int_{-1}^0 [1] dx - \int_0^1 [-x^2 + 1] dx$$

$$0 + \frac{2}{3} - 1 - \frac{2}{3} = \textcolor{red}{-1}$$

$$\int_{\cap}^{\cup} \nabla dx dy = -1$$

$$\oint = \int_{x_3}^{x_2} \int_{\cap}^{\cup} \nabla dx dy = \int_{-1}^1 \int_{\cap}^{\cup} \nabla dx dy = \int_{-1}^1 -1 dx = \textcolor{red}{-2}$$

9. Use Green's theorem to evaluate

$$\oint 6xy dx + x^2 dy$$

where C is the region bounded by $y = 2 - x^2$ and $y = x$

$$\oint 6xydx + x^2dy$$

$$\nabla = \begin{vmatrix} 6xy & x^2 \\ dx & dy \end{vmatrix} = -[6xy]dy + [x^2]dx = -6x + 2x = -4x$$

$$\begin{aligned} & \begin{matrix} y = 2 - x^2 \\ x = 2 - x^2 \end{matrix} \rightarrow \begin{matrix} y = x \\ x^2 + x - 2 = 0 \end{matrix} \\ \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} &= \frac{-(1) \pm \sqrt{(1)^2 - 4(1)(-2)}}{2(1)} \rightarrow x_1 = -2, \quad x_2 = 1 \end{aligned}$$

$$\begin{aligned} \oint &= \int_{x_1}^{x_2} \int_{y_1}^{y_2} \nabla dx dy = \int_{-2}^1 \int_x^{2-x^2} (-4x) dx dy \\ & \int_x^{2-x^2} (-4x) dy = [-8x + 4x^3 + 4x^2] \end{aligned}$$

bounde:

$$\begin{aligned} [-4xy] &= -4x[(2 - x^2) - (x)] = -4x[2 - x^2 - x] = -8x + 4x^3 + 4x^2 \\ & \int_{-2}^1 (-8x + 4x^3 + 4x^2) dx = 9 \end{aligned}$$

bounde:

$$\begin{aligned} &= \left[-\frac{8}{2}x^2 + \frac{4}{4}x^4 + \frac{4}{3}x^3 \right] = \left[-4x^2 + x^4 + \frac{4}{3}x^3 \right] \\ &= \left\{ -4[(1)^2 - (-2)^2] + [(1)^4 - (-2)^4] + \frac{4}{3}[(1)^3 - (-2)^3] \right\} \\ &= \left\{ -4[1 - 4] + [1 - 16] + \frac{4}{3}[1 + 8] \right\} = \left\{ -4[-3] + [-15] + \frac{4}{3}[9] \right\} \\ &= \left[12 - 15 + \frac{36}{3} \right] = 9 \end{aligned}$$

$$\begin{aligned} \oint &= \int_{x_1}^{x_2} \int_{y_1}^{y_2} \nabla dx dy = \int_{-2}^1 \int_x^{2-x^2} (-4x) dx dy \\ & \int_x^{2-x^2} (-4x) dy = [-8x + 4x^3 + 4x^2] \\ & \int_{-2}^1 (-8x + 4x^3 + 4x^2) dx = 9 \end{aligned}$$

$$\oint = \int_{-2}^1 \int_x^{2-x^2} (-4x) dx dy = 9$$

10. Verify Green's theorem for the line integral

$$\oint (x^2 - 2y^2) dx + (x^2 + y^2) dy$$

dy where C is a triangular contour with vertices (0, 0), (4, 2), (2, 2),

$$\oint (x^2 - 2y^2) dx + (x^2 + y^2) dy$$

$$\nabla = \left| \begin{array}{cc} (x^2 - 2y^2) & (x^2 + y^2) \\ dx & dy \end{array} \right| = -[x^2 - 2y^2] dy + [x^2 + y^2] dx$$

$$\nabla = -[-4y] + [2x] = 2x + 4y$$

$(0,0), (4,2), (2,2)$

$$A(0,0) = (x_1, y_1) \rightarrow x_1 = 0, \quad y_1 = 0$$

$$B(4,2) = (x_2, y_2) \rightarrow x_2 = 4, \quad y_2 = 2$$

$$C(2,2) = (x_3, y_3) \rightarrow x_3 = 2, \quad y_3 = 2$$

$(A \rightarrow B)$ vertices $(0,0)(4,2)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 0}{4 - 0} = \frac{1}{2}$$

tangent line

$$y - y_1 = m(x - x_1) \quad \text{Or} \quad y - y_2 = m(x - x_2)$$

$$y - 0 = \frac{1}{2}(x - 0) \quad y - 2 = \frac{1}{2}(x - 4)$$

$$y - 0 = \frac{1}{2}(x) \quad y - 2 = \frac{1}{2}x - 2$$

$$y = \frac{1}{2}x \quad y = \frac{1}{2}x$$

$(B \rightarrow C)$ vertices $(4,2)(2,2)$

$$m = \frac{y_3 - y_2}{x_3 - x_2} = \frac{2 - 2}{2 - 4} = 0$$

tangent line

$$y - y_2 = m(x - x_2) \quad \text{Or} \quad y - y_3 = m(x - x_3)$$

$$y - 2 = 0(x - 4) \quad y - 2 = 0(x - 2)$$

$$y - 2 = 0 \quad y - 2 = 0$$

$$y = 2 \quad y = 2$$

$(C \rightarrow A)$ vertices $(2,2)(0,0)$

$$m = \frac{y_3 - y_1}{x_3 - x_1} = \frac{0 - 2}{0 - 2} = 1$$

tangent line

$$y - y_1 = m(x - x_1) \quad \text{Or} \quad y - y_3 = m(x - x_3)$$

$$y - 2 = 1(x - 2) \quad y - 0 = 1(x - 0)$$

$$y - 2 = x - 2 \quad y = 1(x)$$

$$y = x \quad y = x$$

bounde

$$\int_4^2 \nabla dx \quad y = 2 \quad \int_2^2 \nabla dy = 0$$

$(C, B) \leftarrow dx$
midfielder

(2 < x < 4)

(0 < x < 2)

endpoint
dy → (C, A)
(2 > y > 0)

$\int_2^0 \nabla dy$ $y = x$ $\int_2^x \nabla dy$

dx → (A, B)
starting point
(0 < x < 4)

tangent line $y = \frac{1}{2}x$ $\int_0^4 \nabla dx$ $\int_0^{\frac{1}{2}x} \nabla dy$

(C, A) + (C, B)

y = x tangent line y = 2

(0 < x < 2) dx (2 < x < 4)

x + 2

tangent line add ∪

tangent line $y = \frac{1}{2}x$ (A, B) (0 < x < 4)

$$\oint = \int_{x_1}^{x_3} \int_y^U \nabla dx dy = \int_0^4 \int_y^U \nabla dx dy$$

$$\int \nabla dy = \int [2x + 4y] dy = [2xy + 2y^2] = \text{⌘}$$

$$\int_0^2 (\text{⌘}) dx + \int_2^4 (\text{⌘}) dx - \int_0^4 (\text{⌘}) dx$$

$$\int_0^2 [2x + 4y] dx + \int_2^4 [2x + 4y] dx - \int_0^4 [2x + 4y] dx$$

$$\int_0^4 \leftarrow (A, B) \rightarrow y = \frac{1}{2}x$$

$$[2xy + 2y^2] = \left[2x \left(\frac{1}{2}x \right) + 2 \left(\frac{1}{2}x \right)^2 \right] = \left[x^2 + \frac{1}{2}x^2 \right] = \left[\frac{3}{2}x^2 \right]$$

$$\int_2^4 \leftarrow (C, B) \rightarrow y = 2$$

$$[2xy + 2y^2] = [2x(2) + 2(2)^2] = [4x + 8]$$

$$\int_0^2 \leftarrow (C, A) \rightarrow y = x$$

$$[2xy + 2y^2] = [2x(x) + 2(x)^2] = [2x^2 + 2x^2] = [4x^2]$$

$$\oint = (C, A) + (C, B) - (A, B)$$

green's theorem

$$\oint = \int_{x_1}^{x_3} \int_y^U \nabla \, dx dy = \int_0^4 \int_{\frac{1}{2}x}^{x+2} \nabla \, dx dy$$

integrate

$$\oint = \int_{x_1}^{x_3} \int_y^U \nabla \, dx dy = \int_0^4 \int_y^U \nabla \, dx dy$$

$$\int \nabla \, dy = \int [2x + 4y] \, dy = [2xy + 2y^2] = \aleph$$

$$\int_0^2 (\aleph) dx + \int_2^4 (\aleph) dx - \int_0^4 (\aleph) dx$$

$$\int_0^2 [2xy + 2y^2] dx + \int_2^4 [2xy + 2y^2] dx - \int_0^4 [2xy + 2y^2] dx$$

conversion

$$\int_0^2 (y = x) dx + \int_2^4 (y = 2) dx - \int_0^4 \left(y = \frac{1}{2}x \right) dx$$

$$\int_0^2 [4x^2] dx + \int_2^4 [4x + 8] dx - \int_0^4 \left[\frac{3}{2}x^2 \right] dx$$

$$\frac{32}{3} + 40 - 32 = \frac{56}{3}$$

2.2.2. First Order Differential Equation

3. Solve the following differential equation:

$$\begin{aligned}
 (a) &= \sin t \frac{dy}{dx} + (\cos t)y = \sin^2(t) \\
 &= \sin t \frac{dy}{dx} + \cos t y = \sin^2(t) \\
 &= \frac{\sin t}{\sin t} \frac{dy}{dx} + \frac{\cos t}{\sin t} y = \frac{\sin^2(t)}{\sin t} \\
 &= \frac{dy}{dx} + \frac{\cos t}{\sin t} y = \sin t \\
 \tan \theta &= \frac{\sin t}{\cos t} \quad \cot \theta = \frac{1}{\tan t} = \frac{\cos t}{\sin t} \\
 (a) &= \frac{dy}{dx} + \cot(t)y = \sin t
 \end{aligned}$$

First Order Differential Equations

$$y' + P(t)y = Q(t)$$

$$\mu = e^{\int P(t)dt}$$

$$y\mu = \int \mu \times Q(t)dt$$

$$y' = \frac{dy}{dx}$$

$$(a) = \frac{dy}{dx} + \cot(t)y = \sin t$$

$$P(t) = \cot(t) \quad Q(t) = \sin t$$

$$\mu = e^{\int \cot(t)dt} = e^{\ln \sin t} = \sin t$$

$$y \sin t = \int \sin t \times \sin t dt = \int \sin^2(t)dt$$

$$\cos(a - b) = \cos a \cos b + \sin a \sin b \quad [1]$$

$$\cos(a + a) = \cos a \cos b - \sin a \sin b \quad [2]$$

$$\cos(a - b) - \cos(a + a) = \cos a \cos b + \sin a \sin b - \cos a \cos b + \sin a \sin b \quad [1]$$

- [2]

$$\cos(a - b) - \cos(a + a) = \sin a \sin b + \sin a \sin b$$

$$\cos(a - b) - \cos(a + a) = 2 \sin a \sin b$$

$$\sin a \sin b = \frac{[\cos(a - b) - \cos(a + b)]}{2}$$

when $a = b$ we get a double angles

$$\sin a \sin a = \frac{[\cos(a - a) - \cos(a + a)]}{2}$$

$$\sin^2(a) = \frac{[\cos(0) - \cos(2a)]}{2}$$

$$\sin^2(a) = \frac{[1 - \cos(2a)]}{2} = \frac{1}{2}[1 - \cos(2a)]$$

$$\sin^2(t) = \frac{[1 - \cos(2t)]}{2} = \frac{1}{2}[1 - \cos(2t)]$$

$$y \sin t = \int \sin t \times \sin t dt = \int \sin^2(t) dt = \int \frac{1}{2} [1 - \cos(2t)] dt = \frac{1}{2} \int [1 - \cos(2t)] dt$$

$$y \sin t = \frac{1}{2} \int [1] dt - \frac{1}{2} \int [\cos(2t)] dt = \frac{1}{2} [t] - \frac{1}{2} \left[\frac{\sin(2t)}{2} \right] + C = \frac{1}{2} [t] - \frac{1}{4} [\sin(2t)] + C$$

$$y \sin t = \frac{1}{2} [t] - \frac{1}{4} [\sin(2t)] + C$$

$$y \frac{\sin t}{\sin t} = \frac{\frac{1}{2} [t] - \frac{1}{4} [\sin(2t)] + C}{\sin t}$$

$$y = \frac{\frac{1}{2} [t] - \frac{1}{4} [\sin(2t)] + C}{\sin t} = \frac{\frac{1}{2} [t]}{\sin t} - \frac{\frac{1}{4} [\sin(2t)]}{\sin t} + \frac{C}{\sin t} = \frac{1}{2} \frac{[t]}{\sin t} - \frac{1}{4} \frac{[\sin(2t)]}{\sin t} + \frac{C}{\sin t}$$

$$y = \frac{1}{2} \frac{t}{\sin t} - \frac{1}{4} \frac{\sin(2t)}{\sin t} + \frac{C}{\sin t}$$

$$\sin(2t) = 2 \sin t \cos t$$

$$\sin(a - b) = \cos a \sin b - \sin a \cos b \quad [1]$$

$$\sin(a + a) = \cos a \sin b + \sin a \cos b \quad [2]$$

$$\text{double angles } a = b$$

$$\sin(a - a) = \cos a \sin a - \sin a \cos a \quad [1]$$

$$\sin(0) = 0$$

$$\sin(a + a) = \cos a \sin a + \sin a \cos a \quad [2]$$

$$\sin(2a) = 2 \sin a \cos a$$

$$y = \frac{1}{2} \frac{t}{\sin t} - \frac{1}{4} \frac{\sin(2t)}{\sin t} + \frac{C}{\sin t}$$

$$y = \frac{1}{2} \frac{t}{\sin t} - \frac{1}{4} \frac{[2 \sin t \cos t]}{\sin t} + \frac{C}{\sin t}$$

$$y = \frac{1}{2} \frac{t}{\sin t} - \frac{2}{4} \frac{[\cos t]}{\sin t} + \frac{C}{\sin t}$$

$$y = \frac{1}{2} \frac{t}{\sin t} - \frac{1}{2} \frac{[\cos t]}{\sin t} + \frac{C}{\sin t}$$

$$\csc t = \frac{1}{\sin t}$$

$$\sec t = \frac{1}{\cos t}$$

$$\cot t = \frac{1}{\tan t}$$

$$\tan t = \frac{1}{\cot t}$$

$$\tan \theta = \frac{\sin t}{\cos t}$$

$$\cot \theta = \frac{1}{\tan t} = \frac{\cos t}{\sin t}$$

$$y = \frac{1}{2} \frac{t}{\sin t} - \frac{1}{2} \frac{[\cos t]}{\sin t} + \frac{C}{\sin t}$$

$$y = \frac{1}{2} t \csc t - \frac{1}{2} \cot \theta + C \csc t$$

$$y = \frac{1}{2} t \csc t - \frac{1}{2} \cot \theta + C \csc t$$

$$\begin{aligned}
 (b) &= t^2 \frac{dy}{dt} + 3ty = \frac{\sqrt{1+t^2}}{t^2} \\
 &= \frac{t^2}{t^2} \frac{dy}{dt} + 3 \frac{t}{t^2} y = \frac{\sqrt{1+t^2}}{t^2} \\
 &= \frac{dy}{dt} + 3 \frac{1}{t} y = \frac{\sqrt{1+t^2}}{t^2} \times \frac{1}{t^2} \\
 &= \frac{dy}{dt} + 3 \frac{1}{t} y = \frac{\sqrt{1+t^2}}{t^4}
 \end{aligned}$$

First Order Differential Equations

$$\frac{dy}{dx} + P(t)y = Q(t)$$

$$\mu = e^{\int P(t)dt}$$

$$y\mu = \int \mu \times Q(t)dt$$

$$\begin{aligned}
 &= \frac{dy}{dt} + 3 \frac{1}{t} y = \frac{\sqrt{1+t^2}}{t^4} \\
 P(t) &= 3 \frac{1}{t} \quad Q(t) = \frac{\sqrt{1+t^2}}{t^4} \\
 \mu &= e^{\int 3 \frac{1}{t} dt} = e^{3 \int \frac{1}{t} dt} = e^{3 \ln t} = e^{\ln(t)^3} = (t)^3 = t^3 \\
 yt^3 &= \int t^3 \times \frac{\sqrt{1+t^2}}{t^4} dt = \int \frac{\sqrt{1+t^2}}{t} dt = \int \frac{1}{t} \sqrt{1+t^2} dt
 \end{aligned}$$

$$\begin{aligned}
 (1+t^2) &= u & 2tdt &= udu & dt &= \frac{du}{2t} \\
 2tdt &= u & \frac{t}{\sqrt{(1+t^2)}} &= v
 \end{aligned}$$

$$\begin{aligned}
 \int \frac{1}{t} \sqrt{1+t^2} dt &= \int \frac{1}{t} \sqrt{u} \frac{du}{2t} = \frac{1}{t} \times \frac{1}{2t} \int \sqrt{u} du = \frac{1}{2t^2} \int \sqrt{u} du = \frac{1}{2t^2} \int (u)^{\frac{1}{2}} du \\
 &= \frac{1}{2t^2} \frac{(u)^{\frac{3}{2}}}{\frac{3}{2}} + C \\
 &= \frac{1}{2t^2} \frac{(u)^{\frac{3}{2}}}{\frac{3}{2}} \times \frac{2}{\frac{3}{2}} + C = \frac{\frac{2}{3}}{2t^2} (u)^{\frac{3}{2}} + C = \frac{2}{3} \frac{1}{2t^2} (u)^{\frac{3}{2}} + C = \frac{1}{3t^2} (u)^{\frac{3}{2}} + C = \frac{1}{3t^2} \sqrt{u^3} + C \\
 &= \frac{1}{3t^2} \sqrt{u^3} + C = \frac{1}{3t^2} \sqrt{[1+t^2]^3} + C = \frac{1}{3t^2} \sqrt{[1+t^2][1+t^2]^2} + C \\
 &= \frac{[1+t^2]}{3t^2} \sqrt{[1+t^2]} + C
 \end{aligned}$$

$$\begin{aligned}
&= \left[\frac{1}{3t^2} + \frac{t^2}{3t^2} \right] \sqrt{[1+t^2]} + C = \left[\frac{1}{3t^2} + \frac{1}{3} \right] \sqrt{[1+t^2]} + C \\
&= \left[\frac{1}{3t^2} \right] \sqrt{[1+t^2]} + \left[\frac{1}{3} \right] \sqrt{[1+t^2]} + C \\
&= \frac{1}{3t^2} \sqrt{[1+t^2]} + \frac{1}{3} \sqrt{[1+t^2]} + C = \frac{\sqrt{[1+t^2]}}{3t^2} + \frac{\sqrt{[1+t^2]}}{3} + C \\
&= \frac{\sqrt{1+t^2}}{3t^2} + \frac{\sqrt{1+t^2}}{3} + C \\
&y t^3 = \frac{\sqrt{1+t^2}}{3t^2} + \frac{\sqrt{1+t^2}}{3} + C \\
&y \frac{t^3}{t^3} = \frac{\frac{\sqrt{1+t^2}}{3t^2} + \frac{\sqrt{1+t^2}}{3} + C}{t^3} \\
&y = \frac{\frac{\sqrt{1+t^2}}{3t^2} + \frac{\sqrt{1+t^2}}{3} + C}{t^3} = \frac{\frac{\sqrt{1+t^2}}{3t^2}}{t^3} + \frac{\frac{\sqrt{1+t^2}}{3}}{t^3} + \frac{C}{t^3} = \frac{\sqrt{1+t^2}}{3t^2} \frac{1}{t^3} + \frac{\sqrt{1+t^2}}{3} \frac{1}{t^3} + \frac{C}{t^3} \\
&y = \frac{\sqrt{1+t^2}}{3t^5} + \frac{\sqrt{1+t^2}}{3t^3} + \frac{C}{t^3} = \frac{\sqrt{1+t^2}}{3t^5} + \frac{\sqrt{1+t^2}}{3t^3} + C t^{-3} \\
&= \frac{t^{-5} \sqrt{1+t^2}}{3} + \frac{t^{-3} \sqrt{1+t^2}}{3} + C t^{-3} \\
&y = \frac{t^{-5} \sqrt{1+t^2}}{3} + \frac{t^{-3} \sqrt{1+t^2}}{3} + C t^{-3} = \frac{\sqrt{1+t^2}}{3t^5} + \frac{\sqrt{1+t^2}}{3t^3} + \frac{C}{t^3}
\end{aligned}$$

4. In the circuit shown in Figure 1, a battery supplies a constant voltage of 40V, the inductance is 2H, the resistance is 10Ω , and $I(0) = 0$.

$$\begin{aligned}
E &= V r_1 + V_L \\
E &= I(t)_1 R_1 + L_1 \frac{d}{dx} I(t)_1 \\
40 &= 10I(t)_1 + 2 \frac{d}{dx} I(t)_1 \\
\frac{40}{2} &= \frac{10}{2} I(t)_1 + \frac{2}{2} \frac{d}{dx} I(t)_1 \\
20 &= 5I(t)_1 + \frac{d}{dx} I(t)_1 \\
20 &= \frac{d}{dx} I(t)_1 + 5I(t)_1
\end{aligned}$$

First Order Differential Equations

$$\begin{aligned}
\frac{dy}{dx} + P(t)y &= Q(t) \\
\mu &= e^{\int P(t)dt} \\
y\mu &= \int \mu \times Q(t)dt
\end{aligned}$$

$$y = I(t)_1$$

$$\frac{d}{dx} I(t)_1 + 5I(t)_1 = 20$$

$$P(t) = 5 \quad Q(t) = 20$$

$$\mu = e^{\int 5 dt} = e^{5t}$$

$$ye^{5t} = \int e^{5t} \times 20 dt = 20 \int e^{5t} dt = 20 \left[\frac{e^{5t}}{5} \right] + C = \frac{20}{5} [e^{5t}] + C = 4[e^{5t}] + C$$

$$= 4e^{5t} + C$$

$$ye^{5t} = 4e^{5t} + C$$

$$y \frac{e^{5t}}{e^{5t}} = \frac{4e^{5t} + C}{e^{5t}}$$

$$y \frac{e^{5t}}{e^{5t}} = 4 \frac{e^{5t}}{e^{5t}} + C \frac{1}{e^{5t}}$$

$$y = 4 + C \frac{1}{e^{5t}} = 4 + Ce^{-5t}$$

$$y = I(t)_1 = 4 + Ce^{-5t}$$

$$(a) I(t)_1 = 4 + Ce^{-5t}$$

$$I(t)_1 = ?$$

initial conditions

$$I(0) = 0 \quad t = 0$$

$$I(t)_1 = 4 + Ce^{-5t}$$

$$0 = 4 + Ce^{-5(0)} = 4 + Ce^0 = 4 + C$$

$$C = -4$$

$$I(t)_1 = 4 + Ce^{-5t}$$

$$I(t)_1 = 4 - 4e^{-5t}$$

(b) Find the current after 0.1 seconds. $I(0.1)_1 = ?$

$$I(t)_1 = 4 - 4e^{-5t}$$

$$I(0.1)_1 = ?$$

$$t = 0.1$$

$$I(0.1)_1 = 4 - 4e^{-5(0.1)} = 1.573877361$$

$$I(0.1)_1 = 1.573877361A$$

5. In the circuit shown in Figure 1,
a generator supplies a voltage of $E(t) = 20 \sin(40t)$ volts, the inductance is $0.5H$, the
resistance is 5Ω , and $I(0) = 2A$.

(a) Find $I(t)$

(b) Find the current after 0.1 seconds.

(c) Use a graphing device to draw the graph of the current function.

$$E = Vr_1 + V_L$$

$$\begin{aligned}
E &= I(t)_1 R_1 + L_1 \frac{d}{dx} I(t)_1 \\
20 \sin 40t &= 5I(t)_1 + 0.5 \frac{d}{dx} I(t)_1 \\
\frac{20}{0.5} \sin 40t &= \frac{5}{0.5} I(t)_1 + \frac{0.5}{0.5} \frac{d}{dx} I(t)_1 \\
40 \sin 40t &= 10I(t)_1 + \frac{d}{dx} I(t)_1 \\
40 \sin 40t &= \frac{d}{dx} I(t)_1 + 10I(t)_1 \\
\frac{d}{dx} I(t)_1 + 10I(t)_1 &= 40 \sin 40t
\end{aligned}$$

First Order Differential Equations

$$\frac{dy}{dx} + P(t)y = Q(t)$$

$$\mu = e^{\int P(t)dt}$$

$$y\mu = \int \mu \times Q(t)dt$$

$$y = I(t)_1$$

$$\begin{aligned}
\frac{d}{dx} I(t)_1 + 10I(t)_1 &= 40 \sin 40t \\
P(t) = 10 \quad Q(t) &= 40 \sin 40t \\
\mu = e^{\int 10 dt} &= e^{10t}
\end{aligned}$$

$$\begin{aligned}
ye^{10t} &= \int e^{10t} \times 40 \sin 40t dt = 40 \int e^{10t} [\sin 40t] dt = 40 \int e^{10t} \left[\frac{e^{j40t} - e^{-j40t}}{2j} \right] dt \\
&= \frac{40}{2j} \int [e^{10t} e^{j40t} - e^{10t} e^{-j40t}] dt = \frac{40}{2j} \int [e^{10t+j40t} - e^{10t-j40t}] dt \\
&= \frac{20}{j} \int [e^{10t+j40t} - e^{10t-j40t}] dt = \frac{20}{j} \left[\frac{e^{10t+j40t}}{10+j40} - \frac{e^{10t-j40t}}{10-j40} \right] \\
&= \frac{2}{j} \left[\frac{e^{10t+j40t}}{1+j4} - \frac{e^{10t-j40t}}{1-j4} \right] + C \\
&= 2 \left[\frac{e^{10t+j40t}}{j-4} - \frac{e^{10t-j40t}}{j+4} \right] + C = 2 \left[\frac{e^{10t+j40t}}{j-4} \right] - 2 \left[\frac{e^{10t-j40t}}{j+4} \right] + C \\
&= \frac{2}{j-4} [e^{10t+j40t}] - \frac{2}{j+4} [e^{10t-j40t}] + C = -\frac{8+2j}{17} [e^{10t+j40t}] - \frac{8-2j}{17} [e^{10t-j40t}] + C \\
&= -\frac{8+2j}{17} [e^{j40t} e^{10t}] - \frac{8-2j}{17} [e^{10t} e^{-j40t}] + C \\
&= -e^{10t} \frac{[8+2j]}{17} [e^{j40t}] - e^{10t} \frac{[8-2j]}{17} [e^{-j40t}] + C \\
&= -e^{10t} \left[\frac{8}{17} + \frac{2j}{17} \right] [e^{j40t}] - e^{10t} \left[\frac{8}{17} - \frac{2j}{17} \right] [e^{-j40t}] + C \\
&= -e^{10t} \left[\frac{8}{17} \right] [e^{j40t}] - e^{10t} \left[\frac{2j}{17} \right] [e^{j40t}] - e^{10t} \left[\frac{8}{17} \right] [e^{-j40t}] + e^{10t} \left[\frac{2j}{17} \right] [e^{-j40t}] + C
\end{aligned}$$

$$\begin{aligned}
&= -e^{10t} \left[\frac{8}{17} \right] [e^{j40t}] - e^{10t} \left[\frac{8}{17} \right] [e^{-j40t}] - e^{10t} \left[\frac{2j}{17} \right] [e^{j40t}] + e^{10t} \left[\frac{2j}{17} \right] [e^{-j40t}] + C \\
&= -e^{10t} \left[\frac{8}{17} \right] [e^{j40t} + e^{-j40t}] - e^{10t} \left[\frac{2j}{17} \right] [e^{j40t} - e^{-j40t}] + C \\
&= -e^{10t} \left[\frac{8}{17} \right] [e^{j40t} + e^{-j40t}] - e^{10t} \left[\frac{2j}{17} \right] \left[\frac{j}{j} \right] [e^{j40t} - e^{-j40t}] + C \\
&= -e^{10t} \left[\frac{8}{17} \right] [e^{j40t} + e^{-j40t}] + e^{10t} \left[\frac{2}{17} \right] \left[\frac{1}{j} \right] [e^{j40t} - e^{-j40t}] + C \\
&= -e^{10t} \left[\frac{8}{17} \right] [2 \cos 40t] + e^{10t} \left[\frac{2}{17} \right] [2 \sin 40t] + C \\
&= -e^{10t} \left[\frac{16}{17} \right] [\cos 40t] + e^{10t} \left[\frac{4}{17} \right] [\sin 40t] + C \\
&ye^{10t} = -e^{10t} \left[\frac{16}{17} \right] [\cos 40t] + e^{10t} \left[\frac{4}{17} \right] [\sin 40t] + C \\
&ye^{10t} = -e^{10t} \left[\frac{16}{17} \right] [\cos 40t] + e^{10t} \left[\frac{4}{17} \right] [\sin 40t] + C \\
&y \frac{e^{10t}}{e^{10t}} = \frac{-e^{10t} \left[\frac{16}{17} \right] [\cos 40t] + e^{10t} \left[\frac{4}{17} \right] [\sin 40t] + C}{e^{10t}} \\
&y = \frac{-e^{10t} \left[\frac{16}{17} \right] [\cos 40t] + e^{10t} \left[\frac{4}{17} \right] [\sin 40t] + C}{e^{10t}} \\
&= \frac{-e^{10t} \left[\frac{16}{17} \right] [\cos 40t]}{e^{10t}} + \frac{e^{10t} \left[\frac{4}{17} \right] [\sin 40t]}{e^{10t}} + \frac{C}{e^{10t}} \\
&y = - \left[\frac{16}{17} \right] \frac{e^{10t}}{e^{10t}} [\cos 40t] + \left[\frac{4}{17} \right] \frac{e^{10t}}{e^{10t}} [\sin 40t] + \frac{C}{e^{10t}} \\
&= - \left[\frac{16}{17} \right] [\cos 40t] + \left[\frac{4}{17} \right] [\sin 40t] + \frac{C}{e^{10t}} \\
&y = -\frac{16}{17} [\cos 40t] + \frac{4}{17} [\sin 40t] + \frac{C}{e^{10t}} = -\frac{16}{17} \cos 40t + \frac{4}{17} \sin 40t + Ce^{-10t}
\end{aligned}$$

$$\begin{aligned}
&y = I(t)_1 \\
(a) \quad I(t)_1 &= \frac{4}{17} \sin 40t - \frac{16}{17} \cos 40t + Ce^{-10t}
\end{aligned}$$

$$I(0) = 2A$$

$$I(t)_1 = ?$$

initial conditions

$$I(0) = 2 \quad t = 0$$

$$I(t)_1 = \frac{4}{17} \sin 40t - \frac{16}{17} \cos 40t + Ce^{-10t}$$

$$I(0)_1 = \frac{4}{17} \sin 40(0) - \frac{16}{17} \cos 40(0) + Ce^{-10(0)}$$

$$2 = \frac{4}{17} \sin(0) - \frac{16}{17} \cos(0) + Ce^{(0)}$$

$$2 = \frac{4}{17} (0) - \frac{16}{17} \cos(0) + C(1)$$

$$2 = (0) - \frac{16}{17} (1) + C$$

$$2 = -\frac{16}{17} + C$$

$$C = 2 + \frac{16}{17} = \frac{50}{17}$$

$$C = \frac{50}{17}$$

$$I(t)_1 = \frac{4}{17} \sin 40t - \frac{16}{17} \cos 40t + Ce^{-10t}$$

$$I(t)_1 = \frac{4}{17} \sin 40t - \frac{16}{17} \cos 40t + \frac{50}{17} e^{-10t}$$

(b) Find the current after 0.1 seconds. $I(0.1)_1 = ?$

$$I(t)_1 = \frac{4}{17} \sin 40t - \frac{16}{17} \cos 40t + \frac{50}{17} e^{-10t}$$

$$I(0.1)_1 = \frac{4}{17} \sin 40(0.1) - \frac{16}{17} \cos 40(0.1) + \frac{50}{17} e^{-10(0.1)}$$

$$I(0.1)_1 = \frac{4}{17} \sin(4) - \frac{16}{17} \cos(4) + \frac{50}{17} e^{-(1)}$$

$$I(0.1)_1 = \frac{4}{17} \sin(4) - \frac{16}{17} \cos(4) + \frac{50}{17} e^{-(1)}$$

$$I(0.1)_1 = 0.1595278323A$$

$$I(0.1)_1 \cong 0.1595A$$

(c) Use a graphing device to draw the graph of the current function.

Oups! I'm not going to do it, don't even ask why?

2.2.3 Second order differential equation

$y_{PI} = y$ particular integral

$y_{CF} = y$ complementary function (extra function)

$y =$ general solution

$$y = y_{PI} + y_{CF}$$

1. Consider the following differential equation:

$$y'' - 8y' + 16y = t^2 e^{4t} + 9 \cos(4t)$$

(a) Find the complementary solution of the equation.

$$y'' - 8y' + 16y = t^2 e^{4t} + 9 \cos(4t)$$

$$y[D^2 - 8D + 16] = t^2 e^{4t} + 9 \cos(4t)$$

$$y = 0$$

$$[D^2 - 8D + 16] = 0$$

$$\nabla = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(16)}}{2(1)} \rightarrow D_2 = D_1 = 4$$

$$[D^2 - 8D + 16] = [D - 4][D - 4] = [D - 4]^2$$

$$\text{when } \nabla = m_1 = m_2 \quad y_{PI} = (At + B)e^{mt}$$

$$D_2 = D_1 = 4$$

$$y_{PI} = (At + B)e^{4t}$$

(b) Find the particular solution of the equation.

$$y = y_{PI} + y_{CF}$$

$$y'' - 8y' + 16y = t^2 e^{4t} + 9 \cos(4t)$$

y particular table:

$$t^2 e^{4t} = (At^2 + Bt + C)e^{4t}$$

$$\cos(4t) = A \cos(4t) + B \sin(4t)$$

2. Consider the non homogeneous second order linear equation of the form

$$y'' + 2y' + y = g(t)$$

(a) Find $y_c(t)$, the solution of its corresponding homogeneous equation.

$$y'' + 2y' + y = g(t)$$

$$y[D^2 + 2D + 1] = g(t)$$

$$y = 0$$

$$[D^2 + 2D + 1] = 0$$

$$\nabla = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(1)}}{2(1)} \rightarrow D_2 = D_1 = -1$$

$$[D^2 + 2D + 1] = [D + 1][D + 1] = [D + 1]^2$$

$$\text{when } \nabla = m_1 = m_2 \quad y_{PI} = (At + B)e^{mt}$$

$$D_2 = D_1 = -1$$

$$y_{PI} = (At + B)e^{-t}$$

For each part (b)through (d), find the particular solution of the following equations:

$$(b) y'' + 2y' + y = e^{-t} \sin t$$

$$(c) y'' + 2y' + y = 2te^{-t}$$

$$(d) y'' + 2y' + y = -6t^2 e^t \cos 2t$$

$$(b) y'' + 2y' + y = e^{-t} \sin t$$

$$y_{PI} = (At + B)e^{-t}$$

$$y = y_{PI} + y_{CF}$$

$$y[D^2 + 2D + 1] = e^{-t} \sin t$$

y particular table:

$$e^{-t} \sin t = [A \cos(t) + B \sin(t)]e^{-t}$$

Derivative

$$y = e^{-t} \sin t = [A \cos(t) + B \sin(t)]e^{-t}$$

$$y = [A \cos(t) + B \sin(t)]e^{-t}$$

$$y = [Ae^{-t} \cos(t) + B e^{-t} \sin(t)]$$

$$uv = u'v + uv'$$

$$y' = \{[A \cos(t) + B \sin(t)]e^{-t}\}'$$

$$y' = [A \cos(t) + B \sin(t)]'e^{-t} + [A \cos(t) + B \sin(t)][e^{-t}]'$$

$$y' = [-A \sin(t) + B \cos(t)]e^{-t} - [A \cos(t) + B \sin(t)]e^{-t}$$

$$y' = -A \sin(t)e^{-t} + B \cos(t)e^{-t} - A \cos(t)e^{-t} - B \sin(t)e^{-t}$$

$$y'' = \{[-A \sin(t) + B \cos(t)]e^{-t} - [A \cos(t) + B \sin(t)]e^{-t}\}'$$

$$[-A \sin(t) + B \cos(t)]e^{-t} = [-A \sin(t) + B \cos(t)]'e^{-t} + [-A \sin(t) + B \cos(t)][e^{-t}]'$$

$$= [-A \cos(t) - B \sin(t)]e^{-t} - [-A \sin(t) + B \cos(t)][e^{-t}]'$$

$$= -[A \cos(t) + B \sin(t)][e^{-t}] + [A \sin(t) - B \cos(t)][e^{-t}]'$$

$$= [A \sin(t) - B \cos(t)][e^{-t}] - [A \cos(t) + B \sin(t)][e^{-t}]'$$

$$-[A \cos(t) + B \sin(t)]e^{-t} = -[A \cos(t) + B \sin(t)]'e^{-t} - [A \cos(t) + B \sin(t)][e^{-t}]'$$

$$= [A \sin(t) - B \cos(t)]e^{-t} + [A \cos(t) + B \sin(t)][e^{-t}]'$$

$$y'' = \{[-A \sin(t) + B \cos(t)]e^{-t} - [A \cos(t) + B \sin(t)]e^{-t}\}'$$

$$y'' = [A \sin(t) - B \cos(t)][e^{-t}]' - [A \cos(t) + B \sin(t)][e^{-t}]' + [A \sin(t) - B \cos(t)]e^{-t}$$

$$+ [A \cos(t) + B \sin(t)][e^{-t}]'$$

$$y'' = [A \sin(t) - B \cos(t)][e^{-t}] + [A \sin(t) - B \cos(t)]e^{-t}$$

$$y'' = 2[A \sin(t) - B \cos(t)][e^{-t}] = 2[A \sin(t) - B \cos(t)]e^{-t}$$

$$y'' = 2A \sin(t)e^{-t} - 2B \cos(t)e^{-t}$$

$$\begin{cases} y' = -A \sin(t)e^{-t} + B \cos(t)e^{-t} - A \cos(t)e^{-t} - B \sin(t)e^{-t} \\ y'' = 2A \sin(t)e^{-t} - 2B \cos(t)e^{-t} \\ y = A \cos(t)e^{-t} + B \sin(t)e^{-t} \end{cases}$$

retour

$$y'' + 2y' + y = e^{-t} \sin t$$

$$\begin{aligned}
& [2A \sin(t)e^{-t} - 2B \cos(t)e^{-t}] \\
& + 2[-A \sin(t)e^{-t} + B \cos(t)e^{-t} - A \cos(t)e^{-t} - B \sin(t)e^{-t}] \\
& + [A \cos(t)e^{-t} + B \sin(t)e^{-t}] = e^{-t} \sin t
\end{aligned}$$

$$\begin{aligned}
2A \sin(t)e^{-t} - 2B \cos(t)e^{-t} - 2A \sin(t)e^{-t} + 2B \cos(t)e^{-t} - 2A \cos(t)e^{-t} \\
- 2B \sin(t)e^{-t} + A \cos(t)e^{-t} + B \sin(t)e^{-t} &= e^{-t} \sin t \\
-A \cos(t)e^{-t} - B \sin(t)e^{-t} &= e^{-t} \sin t \\
-B \sin(t)e^{-t} &= e^{-t} \sin t \quad \rightarrow B = -1 \\
-A \cos(t)e^{-t} &= 0 \quad \rightarrow A = 0
\end{aligned}$$

replaced the values in y

$$y = A \cos(t)e^{-t} + B \sin(t)e^{-t} = 0 \cos(t)e^{-t} - \sin(t)e^{-t} = -\sin(t)e^{-t}$$

$$y_{CF} = -\sin(t)e^{-t}$$

$$y = y_{PI} + y_{CF}$$

$$y_{PI} = (At + B)e^{-t}$$

$$y = (At + B)e^{-t} - \sin(t)e^{-t}$$

$$(c) y'' + 2y' + y = 2te^{-t}$$

$$y_{PI} = (At + B)e^{-t}$$

$$y[D^2 + 2D + 1] = 2te^{-t}$$

$$y_{PI} = (At + B)e^{-t}$$

$$y = y_{PI} + y_{CF}$$

$$y[D^2 + 2D + 1] = 2te^{-t}$$

y particular table:

$$2te^{-t} = [ACt + BCt]e^{-t} = Ce^{-t}(At + B)$$

Derivative

$$y = Ce^{-t}(At + B)$$

$$y = Ce^{-t}(At + B)$$

$$y = CAte^{-t} + CBe^{-t}$$

$$uv = u'v + uv'$$

$$y' = [Ce^{-t}(At + B)]' = C[(At + B)e^{-t}]'$$

$$y' = C[(At + B)'e^{-t} - (At + B)(e^{-t})']$$

$$y' = C[Ae^{-t} + (At + B)e^{-t}]$$

$$y' = CAe^{-t} + CAte^{-t} + CBe^{-t}$$

$$y'' = C[Ae^{-t} + (At + B)e^{-t}]'$$

$$[Ae^{-t}]' = -Ae^{-t}$$

$$uv = u'v + uv'$$

$$[(At + B)e^{-t}]' = [Ae^{-t} + (At + B)e^{-t}]'$$

$$y'' = C[Ae^{-t} + (At + B)e^{-t}]' = -CAe^{-t} + CAte^{-t} + C(At + B)e^{-t}$$

$$y'' = C(At + B)e^{-t}$$

$$y'' = CAte^{-t} + CBe^{-t}$$

$$\begin{cases} y' = C[Ae^{-t} + (At + B)e^{-t}] \\ y'' = C(At + B)e^{-t} \\ y = C(At + B)e^{-t} \end{cases}$$

Or

$$\begin{cases} y' = CAe^{-t} + CAte^{-t} + CBe^{-t} \\ y'' = CAte^{-t} + CBe^{-t} \\ y = CAte^{-t} + CBe^{-t} \end{cases}$$

$$\begin{aligned} y'' + 2y' + y &= 2te^{-t} \\ [CAte^{-t} + CBe^{-t}] + 2[CAe^{-t} + CAte^{-t} + CBe^{-t}] + [CAte^{-t} + CBe^{-t}] &= 2te^{-t} \\ CAte^{-t} + CBe^{-t} + 2CAe^{-t} + 2CAte^{-t} + 2CBe^{-t} + CAte^{-t} + CBe^{-t} &= 2te^{-t} \\ 2CAe^{-t} + 4CAte^{-t} + 4CBe^{-t} &= 2te^{-t} \\ [2CA + 4CB]e^{-t} + 4CAte^{-t} &= 2te^{-t} \end{aligned}$$

$$4CAte^{-t} = 2te^{-t} \quad \rightarrow 4CA = 2 \quad \rightarrow CA = \frac{1}{2}$$

$$[2CA + 4CB]e^{-t} = 2te^{-t} \quad \rightarrow [2CA + 4CB] = 0 \quad \rightarrow [1 + 4CB] = 0 \quad \rightarrow 4CB = -1$$

$$\rightarrow CB = -\frac{1}{4}$$

replaced the values in y

$$y = CAte^{-t} + CBe^{-t} = \frac{1}{2}te^{-t} + \left[-\frac{1}{4}\right]e^{-t} = \frac{1}{2}te^{-t} - \frac{1}{4}e^{-t}$$

$$y_{CF} = \frac{1}{2}te^{-t} - \frac{1}{4}e^{-t}$$

$$y = y_{PI} + y_{CF}$$

$$y_{PI} = (At + B)e^{-t}$$

$$y = (At + B)e^{-t} + \frac{1}{2}te^{-t} - \frac{1}{4}e^{-t} = Ate^{-t} + Be^{-t} + \frac{1}{2}te^{-t} - \frac{1}{4}e^{-t}$$

$$y = Ate^{-t} + Be^{-t} + \frac{1}{2}te^{-t} - \frac{1}{4}e^{-t}$$

$$(d) y'' + 2y' + y = -6t^2e^t \cos 2t$$

$$y_{PI} = (At + B)e^{-t}$$

$$y[D^2 + 2D + 1] = -6t^2e^t \cos 2t$$

$$y = y_{PI} + y_{CF}$$

y particular table:

$$-6t^2e^t \cos 2t = [At^2 + Bt + C][A \cos(3t) + B \sin(3t)]e^t$$

2.2.4 Laplace transform

2.2.4 Laplace transform

1. If $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \frac{9}{s^2(s^2+9)}$, find $\mathcal{L}\{e^{2t}f(5t)\}$
2. In the electric circuit shown below

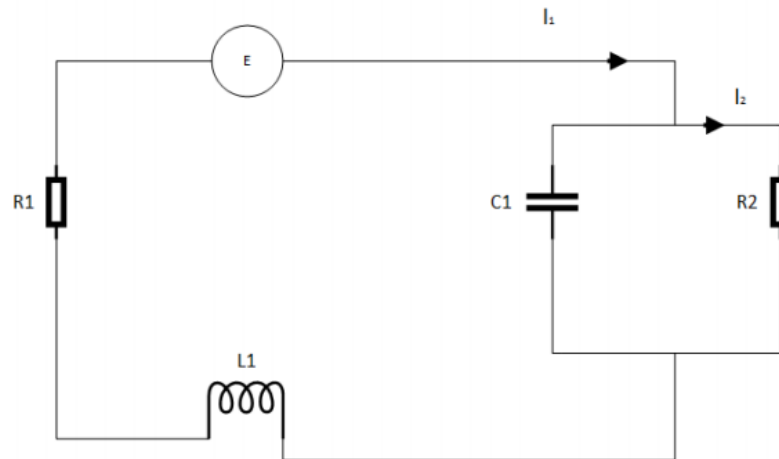
$$E = 500 \sin 10t$$

$$R_1 = 10\Omega$$

$$R_2 = 10\Omega$$

$$L = 1H$$

$$C = 0.01F$$



If the charge on the capacitor and the currents I_1 and I_2 are zero at $t = 0$, find the charge on the capacitor at time $t > 0$

$$E = V_{r_1} + V_{L_1} + V_{c_1}$$

$$0 = V_{r_2} - V_{c_1}$$

$$I(t)_1 - I(t)_2 = I_c$$

$$E = V_{r_1} + V_{L_1} + V_{c_1}$$

$$E = I(t)_1 R_1 + L \frac{dI(t)_1}{dt} + \frac{1}{C} \int I_c dt$$

$$500 \sin 10t = 10I(t)_1 + 1 \frac{dI(t)_1}{dt} + \frac{1}{0.01} \int I_c dt$$

$$500 \sin 10t = 10I(t)_1 + \frac{dI(t)_1}{dt} + 100 \int I_c dt$$

$$\mathcal{L}\{500 \sin 10t\} = \mathcal{L}\left\{10I(t)_1 + \frac{dI(t)_1}{dt} + 100 \int [I(t)_1 - I(t)_2] dt\right\}$$

$$500 \left[\frac{10}{s^2 + 10^2} \right] = 10I(s)_1 + sI(s)_1 + \frac{100}{s} [I(s)_1 - I(s)_2]$$

$$\begin{aligned}
500 \left[\frac{10s}{s^2 + 10^2} \right] &= 10sI(s)_1 + s^2I(s)_1 + 100[I(s)_1 - I(s)_2] \\
\left[\frac{5000s}{s^2 + 10^2} \right] &= 10sI(s)_1 + s^2I(s)_1 + 100I(s)_1 - 100I(s)_2 \\
\left[\frac{5000s}{s^2 + 10^2} \right] &= [10s + s^2 + 100]I(s)_1 - 100I(s)_2 \\
\left[\frac{5000s}{s^2 + 10^2} \right] &= [s^2 + 10s + 100]I(s)_1 - 100I(s)_2
\end{aligned}$$

$$\begin{aligned}
0 &= V_{r_2} - V_{c_1} \\
0 &= I(t)_2 R_2 - \frac{1}{C} \int I_c dt \\
0 &= 10I(t)_2 - \frac{1}{0.01} \int I_c dt \\
0 &= 10I(t)_2 - 100 \int I_c dt \\
0 &= 10I(t)_2 - 100 \int [I(t)_1 - I(t)_2] dt \\
L\{0\} &= L\left\{10I(t)_2 - 100 \int [I(t)_1 - I(t)_2] dt\right\} \\
0 &= 10I(s)_2 - \frac{100}{s} [I(s)_1 - I(s)_2] \\
0 &= 10sI(s)_2 - 100[I(s)_1 - I(s)_2] \\
0 &= 10sI(s)_2 - 100I(s)_1 + 100I(s)_2 \\
0 &= [10s + 100]I(s)_2 - 100I(s)_1 \\
0 &= -100I(s)_1 + [10s + 100]I(s)_2 \\
0 &= -10[10]I(s)_1 + 10[s + 10]I(s)_2 \\
\frac{0}{[10]} &= -[10]I(s)_1 + [s + 10]I(s)_2 \\
0 &= -10I(s)_1 + [s + 10]I(s)_2 \\
0 &= 10I(s)_1 - [s + 10]I(s)_2 \\
0 &= 10I(s)_1 - [s + 10]I(s)_2
\end{aligned}$$

$$\begin{aligned}
\left\{ \left[\frac{5000s}{s^2 + 10^2} \right] \right. &= [s^2 + 10s + 100]I(s)_1 - 100I(s)_2 \\
0 &= 10I(s)_1 - [s + 10]I(s)_2 \\
\Delta &= \begin{vmatrix} [s^2 + 10s + 100] & -100 \\ 10 & -[s + 10] \end{vmatrix} \\
\Delta &= -[s^2 + 10s + 100][s + 10] + 10 \times 100 \\
\Delta &= -[s^3 + 10s^2 + 10s^2 + 100s + 100s + 1000] + 1000
\end{aligned}$$

$$\begin{aligned}\nabla &= -[s^3 + 20s^2 + 200s + 1000] + 1000 \\ \nabla &= -s^3 - 20s^2 - 200s - 1000 + 1000 \\ \nabla &= -s^3 - 20s^2 - 200s \\ \nabla &= -s[s^2 + 20s + 200]\end{aligned}$$

$$\begin{aligned}\nabla_{I_1} &= \begin{vmatrix} \left[\frac{5000s}{s^2 + 10^2} \right] & -100 \\ 0 & -[s + 10] \end{vmatrix} \\ \nabla_{I_1} &= -\left[\frac{5000s}{s^2 + 10^2} \right] \times [s + 10] = -\left[\frac{5000s[s + 10]}{s^2 + 10^2} \right] = -s \times \left[\frac{5000[s + 10]}{s^2 + 10^2} \right]\end{aligned}$$

$$\begin{aligned}\nabla_{I_2} &= \begin{vmatrix} [s^2 + 10s + 100] & \left[\frac{5000s}{s^2 + 10^2} \right] \\ 10 & 0 \end{vmatrix} \\ \nabla_{I_2} &= -10 \times \left[\frac{5000s}{s^2 + 10^2} \right] = -s \times \left[\frac{50000}{s^2 + 10^2} \right] \\ I(s)_1 &= \frac{\nabla_{I_1}}{\nabla} = \frac{-s \times \left[\frac{5000[s + 10]}{s^2 + 10^2} \right]}{-s[s^2 + 20s + 200]} = \frac{\left[\frac{5000[s + 10]}{s^2 + 10^2} \right]}{[s^2 + 20s + 200]} \\ I(s)_1 &= \frac{5000[s + 10]}{[s^2 + 10^2][s^2 + 20s + 200]} \\ &= \frac{5000s + 50000}{[s^2 + 10^2][s^2 + 20s + 200]} \\ &= \frac{5000s}{[s^2 + 10^2][s^2 + 20s + 200]} + \frac{50000}{[s^2 + 10^2][s^2 + 20s + 200]}\end{aligned}$$

$$\begin{aligned}I(s)_2 &= \frac{\nabla_{I_2}}{\nabla} = \frac{-s \times \left[\frac{50000}{s^2 + 10^2} \right]}{-s[s^2 + 20s + 200]} = \frac{\left[\frac{50000}{s^2 + 10^2} \right]}{[s^2 + 20s + 200]} \\ &= \frac{50000}{[s^2 + 10^2][s^2 + 20s + 200]}\end{aligned}$$

$$\begin{aligned}I(s)_c &= \frac{I(s)_1}{I(s)_2} = \frac{I(s)_1}{I(s)_2} \\ I(s)_c &= \frac{5000s}{[s^2 + 10^2][s^2 + 20s + 200]} + \frac{50000}{[s^2 + 10^2][s^2 + 20s + 200]} \\ &\quad - \frac{50000}{[s^2 + 10^2][s^2 + 20s + 200]} \\ I(s)_c &= \frac{5000s}{[s^2 + 10^2][s^2 + 20s + 200]}\end{aligned}$$

now, the laplace forms of the Current in the capacitor is:

$$I(s)_c = \frac{5000s}{[s^2 + 10^2][s^2 + 20s + 200]}$$

the inverse of $I(s)_c$ will be $I(t)_c$

the partial fractions:

$$I(s)_c = \frac{5000s}{[s^2 + 10^2][s^2 + 20s + 200]}$$

$$[s^2 + 10^2] = [s^2 + 100] = [s^2 + 0s + 100] = 0$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{0 \pm \sqrt{0 - 4 \times 100}}{2} = \pm 10j$$

$$s_1 = -10j$$

$$s_2 = +10j$$

$$a^2 + b^2 = (a + bj)(a - bj) \quad a^2 + b^2 = (a + b)^2 - 2ab \quad a^2 + b^2 = (a - b)^2 + 2ab$$

$$[s \pm 10j] = [s^2 + 100] = [s^2 + 0s + 100] = [s + 10j][s - 10j]$$

$$= [(s) + (10j)][(s) - (10j)]$$

$$= [(s)(s) + (s)(10j) - (s)(10j) - (10j)(10j)]$$

$$= [(s)^2 - (10j)^2] = [s^2 + 100] = [s^2 + 10^2]$$

$$[s^2 + 20s + 200] = 0$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-20 \pm \sqrt{20^2 - 4 \times 200}}{2} = -10 \pm 10j$$

$$s_1 = -10 - 10j$$

$$s_2 = -10 + 10j$$

$$a^2 + b^2 = (a + bj)(a - bj) \quad a^2 + b^2 = (a + b)^2 - 2ab \quad a^2 + b^2 = (a - b)^2 + 2ab$$

$$[s + 10 \pm 10j] = [s^2 + 20s + 200] = [s + 10 - 10j][s + 10 + 10j]$$

$$= [(s + 10) - 10j][(s + 10) + 10j]$$

$$= [(s + 10)(s + 10) + (s + 10)(10j) - (s + 10)(10j) - (10j)(10j)]$$

$$= [(s + 10)(s + 10) - (10j)(10j)] = [(s + 10)^2 - (10j)^2]$$

$$= [(s + 10)^2 + (10)^2] = [(s + 10)^2 + 100]$$

$$I(s)_c = \frac{5000s}{[s^2 + 10^2][s^2 + 20s + 200]}$$

$$= \frac{As + B}{[s^2 + 10^2]} + \frac{Cs + D}{[s^2 + 20s + 200]}$$

$$p_t = p_1 + p_2$$

$$p_t = \frac{As + B}{[s^2 + 10^2]} + \frac{Cs + D}{[s^2 + 20s + 200]}$$

Partial I

$$p_1 = \frac{5000s}{[s^2 + 10^2]} = \frac{As + B}{[s^2 + 10^2]} = \frac{As + B}{[s + 10j][s - 10j]}$$

$$p_1 = \frac{A}{[s + 10j]} + \frac{B}{[s - 10j]} = \frac{As + B}{[s^2 + 10^2]}$$

$$5000s = A[s - 10j] + B[s + 10j]$$

$$\text{when } s = -10j \text{ we get } 5000(-10j)$$

$$= A[(-10j) - 10j] + B[(-10j) + 10j]$$

$$5000(-10j) = A[-10j - 10j] + B[-10j + 10j]$$

$$-50000j = A[-20j] + B[0]$$

$$-50000j = -20jA$$

$$50000 = 20A$$

$$A = 2500$$

$$5000s = A[s - 10j] + B[s + 10j]$$

$$\text{when } s = 10j \text{ we get } 5000(10j) = A[(10j) - 10j] + B[(10j) + 10j]$$

$$5000(10j) = A[10j - 10j] + B[10j + 10j]$$

$$50000j = A[0] + B[20j]$$

$$50000j = 20jB$$

$$50000 = 20B$$

$$B = 2500$$

$$p_1 = \frac{5000s}{[s^2 + 10^2]} = \frac{As + B}{[s^2 + 10^2]} = \frac{As + B}{[s + 10j][s - 10j]}$$

$$p_1 = \frac{A}{[s + 10j]} + \frac{B}{[s - 10j]} = \frac{2500}{[s + 10j]} + \frac{2500}{[s - 10j]}$$

$$p_1 = \frac{As + B}{[s^2 + 10^2]} = \frac{2500s + 2500}{[s^2 + 10^2]} = \frac{2500s}{[s^2 + 10^2]} + \frac{2500}{[s^2 + 10^2]}$$

$$e^{bt} = \frac{1}{s - b}$$

$$\sin at = \frac{a}{s^2 + a^2}$$

$$\cos at = \frac{s}{s^2 + a^2}$$

$$e^{bt} \sin at = \frac{a}{(s - b)^2 + a^2}$$

$$e^{bt} \cos at = \frac{s - b}{(s - b)^2 + a^2}$$

the inverse laplace of p_1

$$p_1 = \frac{2500s}{[s^2 + 10^2]} + \frac{250(10)}{[s^2 + 10^2]} = 2500 \cos 10t + 250 \sin 10t$$

or

$$p_1 = \frac{2500}{[s + 10j]} + \frac{2500}{[s - 10j]} = 2500 e^{j10t} + 2500 e^{-j10t}$$

$$p_1 = 2500[e^{j10t} + e^{-j10t}] = 1250 \cos 10t$$

Partial II

$$p_2 = \frac{5000s}{[s^2 + 20s + 200]} = \frac{Cs + D}{[s^2 + 20s + 200]} = \frac{Cs + D}{[s + 10 - 10j][s + 10 + 10j]}$$

$$p_2 = \frac{Cs + D}{[s + 10 - 10j]} + \frac{Cs + D}{[s + 10 + 10j]} = \frac{Cs + D}{[(s + 10)^2 + (10)^2]}$$

$$5000s = C[s + 10 + 10j] + D[s + 10 - 10j]$$

when $s = -10 + 10j$ we get

$$5000(-10 + 10j) = C[(-10 + 10j) + 10 + 10j] + D[(-10 + 10j) + 10 - 10j]$$

$$50000(j - 1) = C[-10 + 10j + 10 + 10j] + D[-10 + 10j + 10 - 10j]$$

$$50000(j - 1) = C[20j] + D[0]$$

$$50000(j - 1) = [20j]C$$

$$\frac{50000(j - 1)}{20j} = C$$

$$C = 2500 \times \frac{(j - 1)}{j} = 2500 \times \left[\frac{j}{j} - \frac{1}{j} \right] = 2500 \times \left[1 - \frac{1}{j} \right] = 2500 \times [1 + j]$$

$$C = 2500[1 + j]$$

$$5000s = C[s + 10 + 10j] + D[s + 10 - 10j]$$

when $s = -10 - 10j$ we get

$$5000(-10 - 10j) = C[(-10 - 10j) + 10 + 10j] + D[(-10 - 10j) + 10 - 10j]$$

$$-50000(j + 1) = C[-10 - 10j + 10 + 10j] + D[-10 - 10j + 10 - 10j]$$

$$-50000(j + 1) = C[0] + D[-20j]$$

$$-50000(j + 1) = [-20j]D$$

$$\frac{50000(j + 1)}{20j} = D$$

$$D = 2500 \times \frac{(j + 1)}{j} = 2500 \times \left[\frac{j}{j} + \frac{1}{j} \right] = 2500 \times \left[1 + \frac{1}{j} \right] = 2500 \times [1 - j]$$

$$D = 2500[1 - j]$$

$$p_2 = \frac{5000s}{[s^2 + 20s + 200]} = \frac{Cs + D}{[s^2 + 20s + 200]} = \frac{Cs + D}{[s + 10 - 10j][s + 10 + 10j]}$$

$$p_2 = \frac{Cs + D}{[s + 10 - 10j]} + \frac{Cs + D}{[s + 10 + 10j]} = \frac{Cs + D}{[(s + 10)^2 + (10)^2]}$$

$$p_2 = \frac{Cs + D}{[(s + 10)^2 + (10)^2]} = \frac{2500[1 + j]s + 2500[1 - j]}{[(s + 10)^2 + (10)^2]}$$

$$p_2 = \frac{2500[1+j]s}{[(s+10)^2 + (10)^2]} + \frac{2500[1-j]}{[(s+10)^2 + (10)^2]}$$

$$p_2 = \frac{C}{[s+10-10j]} + \frac{D}{[s+10+10j]} = \frac{2500[1+j]}{[s+10-10j]} + \frac{2500[1-j]}{[s+10+10j]}$$

$$\sin at = \frac{a}{s^2 + a^2} \quad \cos at = \frac{s}{s^2 + a^2}$$

$$e^{bt} = \frac{1}{s-b} \quad e^{bt} \sin at = \frac{a}{(s-b)^2 + a^2} \quad e^{bt} \cos at = \frac{s-b}{(s-b)^2 + a^2}$$

$$[1+j] = 2 < \frac{\pi}{4} = 2e^{j\frac{\pi}{4}}$$

the inverse laplace of p_2

$$p_2 = \frac{2500[1+j]s}{[(s+10)^2 + (10)^2]} + \frac{2500[1-j]}{[(s+10)^2 + (10)^2]}$$

$$= \frac{2500[1+j]s}{[(s+10)^2 + (10)^2]} + \frac{250[1-j] \times [10]}{[(s+10)^2 + (10)^2]}$$

$$p_2 = 2500[1+j]e^{-10t} \cos 10t + 250[1-j]e^{-10t} \sin 10t$$

$$p_2 = 2500 \left[2e^{j\frac{\pi}{4}} \right] e^{-10t} \cos 10t + 250 \left[2e^{-j\frac{\pi}{4}} \right] e^{-10t} \sin 10t$$

$$p_2 = 5000 \left[e^{j\frac{\pi}{4}} \right] e^{-10t} \cos 10t + 500 \left[e^{-j\frac{\pi}{4}} \right] e^{-10t} \sin 10t$$

$$p_2 = 5000[e^{j45^\circ}]e^{-10t} \cos 10t + 500[e^{-j45^\circ}]e^{-10t} \sin 10t$$

Or

$$p_2 = \frac{2500[1+j]}{[s+10-10j]} + \frac{2500[1-j]}{[s+10+10j]}$$

$$= \frac{2500[1+j]}{[s+(10-10j)]} + \frac{2500[1-j]}{[s+(10+10j)]}$$

$$p_2 = 2500[1+j]e^{-(10-10j)t} + 2500[1-j]e^{-(10+10j)t}$$

$$p_2 = 2500 \left[2e^{j\frac{\pi}{4}} \right] e^{-(10-10j)t} + 2500 \left[2e^{-j\frac{\pi}{4}} \right] e^{-(10+10j)t}$$

$$= 2500 \left[2e^{j\frac{\pi}{4}} \right] [e^{-(10-10j)t}] + 2500 \left[2e^{-j\frac{\pi}{4}} \right] [e^{-(10+10j)t}]$$

$$= 2500 \left[2e^{j\frac{\pi}{4}} \right] [e^{-10t+10jt}] + 2500 \left[2e^{-j\frac{\pi}{4}} \right] [e^{-10t-10jt}]$$

$$= 2500 \left[2e^{j\frac{\pi}{4}} \right] [e^{-10t} e^{10jt}] + 2500 \left[2e^{-j\frac{\pi}{4}} \right] [e^{-10t} e^{-10jt}]$$

$$= 5000 \left[e^{j\frac{\pi}{4}} \right] [e^{-10t} e^{10jt}] + 5000 \left[e^{-j\frac{\pi}{4}} \right] [e^{-10t} e^{-10jt}]$$

$$= 5000[e^{-10t}] \left[e^{j\frac{\pi}{4}} e^{10jt} \right] + 5000[e^{-10t}] \left[e^{-j\frac{\pi}{4}} e^{-10jt} \right]$$

$$= 5000[e^{-10t}] \left[e^{j\frac{\pi}{4}+10jt} \right] + 5000[e^{-10t}] \left[e^{-j\frac{\pi}{4}-10jt} \right]$$

$$= 5000[e^{-10t}] \left[e^{j(\frac{\pi}{4}+10t)} \right] + 5000[e^{-10t}] \left[e^{-j(\frac{\pi}{4}+10t)} \right]$$

$$\begin{aligned}
&= 5000[e^{-10t}] \left\{ \left[e^{j(\frac{\pi}{4}+10t)} \right] + \left[e^{-j(\frac{\pi}{4}+10t)} \right] \right\} \\
&= 5000[e^{-10t}] \left\{ e^{j(\frac{\pi}{4}+10t)} + e^{-j(\frac{\pi}{4}+10t)} \right\} \\
&= 2500[e^{-10t}] \left\{ \cos\left(\frac{\pi}{4} + 10t\right) \right\} = 2500[e^{-10t}] \{\cos(45^\circ + 10t)\} \\
&= 2500[e^{-10t}] \{\cos(45^\circ + 10t)\}
\end{aligned}$$

$$\begin{aligned}
\cos(a+b) &= \cos a \times \cos b - \sin a \times \sin b \\
\cos(45^\circ + 10t) &= \cos(45^\circ) \times \cos(10t) - \sin(45^\circ) \times \sin(10t) \\
\cos\left(\frac{\pi}{4} + 10t\right) &= \cos\left(\frac{\pi}{4}\right) \times \cos(10t) - \sin\left(\frac{\pi}{4}\right) \times \sin(10t) \\
\cos\left(\frac{\pi}{4} + 10t\right) &= \left[\frac{\sqrt{2}}{2}\right] \times \cos(10t) - \left[\frac{\sqrt{2}}{2}\right] \times \sin(10t)
\end{aligned}$$

$$\begin{aligned}
p_2 &= 2500[e^{-10t}] \{\cos(45^\circ + 10t)\} \\
p_2 &= 2500[e^{-10t}] \left\{ \left[\frac{\sqrt{2}}{2}\right] \times \cos(10t) - \left[\frac{\sqrt{2}}{2}\right] \times \sin(10t) \right\} \\
p_2 &= 2500[e^{-10t}] \left\{ \left[\frac{\sqrt{2}}{2}\right] \times \cos(10t) \right\} - 2500[e^{-10t}] \left\{ \left[\frac{\sqrt{2}}{2}\right] \times \sin(10t) \right\} \\
p_2 &= 2500 \left[\frac{\sqrt{2}}{2}\right] [e^{-10t}] \{\cos(10t)\} - 2500 \left[\frac{\sqrt{2}}{2}\right] [e^{-10t}] \{\sin(10t)\} \\
p_2 &= 1250[\sqrt{2}][e^{-10t}] \{\cos(10t)\} - 1250[\sqrt{2}][e^{-10t}] \{\sin(10t)\} \\
p_2 &= 1768[e^{-10t}] \{\cos(10t)\} - 1768[e^{-10t}] \{\sin(10t)\} \\
p_2 &= 1768[e^{-10t}] \cos(10t) - 1768[e^{-10t}] \sin(10t)
\end{aligned}$$

$$p_2 = 1768[e^{-10t}][\cos(10t) - \sin(10t)]$$

$$\begin{aligned}
p_1 &= 2500 \cos 10t + 250 \sin 10t \\
p_2 &= 1768[e^{-10t}] \cos(10t) - 1768[e^{-10t}] \sin(10t)
\end{aligned}$$

$$\begin{aligned}
p_t &= p_1 + p_2 \\
p_t &= 2500 \cos 10t + 250 \sin 10t + 1768[e^{-10t}][\cos(10t) - \sin(10t)]
\end{aligned}$$

the current in the capacitor

$$I(t)_C = 2500 \cos 10t + 250 \sin 10t + 1768[e^{-10t}][\sin(10t) - \cos(10t)]$$

$$I(t)_C = 2500 \cos 10t + 250 \sin 10t + 1768[e^{-10t}] \cos(10t) - 1768[e^{-10t}] \sin(10t)$$

the charge in the capacitor

$$\frac{dq(t)}{dt} = I(t)_C$$

$$q(t) = \int I(t)_C dt$$

$$q(t) = \int [2500 \cos 10t + 250 \sin 10t + 1768[e^{-10t}][\cos(10t) - \sin(10t)]] dt$$

$$q(t) = \int [2500 \cos 10t + 250 \sin 10t + 1768[e^{-10t}] \cos(10t) - 1768[e^{-10t}] \sin(10t)] dt$$

$$q(t) = \int 2500 \cos 10t dt + \int 250 \sin 10t dt - \int 1768[e^{-10t}] \sin(10t) dt + \int 1768[e^{-10t}] \cos(10t) dt$$

$$q(t) = 2500 \int \cos 10t dt + 250 \int \sin 10t dt - 1768 \int [e^{-10t}] \sin(10t) dt + 1768 \int [e^{-10t}] \cos(10t) dt$$

$$\int \cos 10t dt = \frac{1}{10} [\sin 10t]$$

$$\int \sin 10t dt = -\frac{1}{10} [\cos 10t]$$

$$\int [e^{-10t}] \sin(10t) dt = -\frac{1}{20} e^{-10t} [\sin(10t) + \cos(10t)]$$

$$\int [e^{-10t}] \cos(10t) dt = \frac{1}{20} e^{-10t} [\sin(10t) - \cos(10t)]$$

$$q(t) = 2500 \int \cos 10t dt + 250 \int \sin 10t dt - 1768 \int [e^{-10t}] \sin(10t) dt + 1768 \int [e^{-10t}] \cos(10t) dt$$

$$q(t) = \frac{2500}{10} [\sin 10t] - \frac{250}{10} [\cos 10t] + \frac{1768}{20} e^{-10t} [\sin(10t) + \cos(10t)] \\ - \frac{1768}{20} e^{-10t} [\sin(10t) - \cos(10t)]$$

$$q(t) = 250[\sin 10t] - 25[\cos 10t] + \frac{442}{5} e^{-10t} [\sin(10t) + \cos(10t)] \\ - \frac{442}{5} e^{-10t} [\sin(10t) - \cos(10t)]$$

$$q(t) = 250[\sin 10t] - 25[\cos 10t] + 88.4 e^{-10t} [\sin(10t) + \cos(10t)] \\ - 88.4 e^{-10t} [\sin(10t) - \cos(10t)]$$

$$q(t) = 250[\sin 10t] - 25[\cos 10t] + 88.4 e^{-10t} [\sin(10t)] \\ + 88.4 e^{-10t} [\cos(10t)] - 88.4 e^{-10t} [\sin(10t)] - 88.4 e^{-10t} [-\cos(10t)]$$

$$q(t) = 250[\sin 10t] - 25[\cos 10t] + 88.4 e^{-10t} [\sin(10t)] \\ + 88.4 e^{-10t} [\cos(10t)] - 88.4 e^{-10t} [\sin(10t)] + 88.4 e^{-10t} [\cos(10t)]$$

$$q(t) = 250[\sin 10t] - 25[\cos 10t] - \frac{884}{5} e^{-10t} [\cos(10t)]$$

$$q(t) = 250[\sin 10t] - 25[\cos 10t] - 176.8 e^{-10t} [\cos(10t)]$$

$$q(t) = -[\cos(10t)][25 + 176.8 e^{-10t}] + 250[\sin 10t]$$

$$q(t) = -[25 + 176.8 e^{-10t}][\cos(10t)] + 250[\sin 10t]$$

$$q(t) = 250[\sin 10t] - [25 + 176.8 e^{-10t}][\cos(10t)]$$

2.2.5 Fourier series

3. Consider the Fourier series (of period 10) representing
 4. Consider the periodic square wave $x(t)$ shown in figure 2 below:
 (a) Determine the complex exponential Fourier series of $x(t)$.
 (b) Determine the trigonometric Fourier series of $x(t)$

Figure 2: Periodic square wave

- (a) Determine the complex exponential Fourier series of $x(t)$.

$$x(x) = C_0 + \sum_{n=1}^{\infty} C_n e^{j\omega t}$$

$$C_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-jn\omega t} dt$$

$$C_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) e^{-jn\omega t} dt = \frac{1}{T_0} \int_0^{\frac{T_0}{2}} A e^{-jn\omega t} dt + \frac{1}{T_0} \int_{-\frac{T_0}{2}}^0 0 e^{-jn\omega t} dt$$

$$\omega = \frac{2\pi}{P} = \frac{2\pi}{T_0}$$

$$C_n = \frac{1}{T_0} \int_0^{\frac{T_0}{2}} A e^{-jn\omega t} dt + \frac{1}{T_0} \int_{-\frac{T_0}{2}}^0 0 e^{-jn\omega t} dt = \frac{1}{T_0} \int_0^{\frac{T_0}{2}} A e^{-jn\omega t} dt$$

$$C_n = \frac{A}{T_0} \int_0^{\frac{T_0}{2}} e^{-jn\omega t} dt = \frac{A}{T_0} \left[\frac{e^{-jn\omega t}}{-j\omega} \right] = \frac{A}{T_0} \left[\frac{e^{-jn\omega \frac{T_0}{2}} - e^0}{-j\omega} \right] = \frac{A}{T_0} \left[\frac{e^{-jn\frac{2\pi}{T_0} \times \frac{T_0}{2}} - e^0}{-j\omega} \right]$$

$$= \frac{A}{T_0} \left[\frac{e^{-jn\frac{2\pi}{T_0} \times \frac{T_0}{2}} - e^0}{-j\omega} \right]$$

$$= \frac{A}{T_0} \left[\frac{e^{-jn\pi} - 1}{-j\omega} \right] = \frac{A}{-j\omega T_0} [e^{-jn\pi} - 1] = \frac{A}{-j\frac{2\pi}{T_0} T_0} [e^{-jn\pi} - 1] = \frac{A}{-j2\pi} [e^{-jn\pi} - 1]$$

$$= \frac{A}{-j2\pi} [\cos(n\pi) - j \sin(n\pi) - 1] = \frac{A}{-j2\pi} [\cos(n\pi) - 1] = j \frac{A}{2\pi} [\cos(n\pi) - 1]$$

$$C_n = j \frac{A}{2\pi} [(-1)^2 - 1]$$

$$C_0 = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) dt$$

$$C_0 = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) dt = \frac{1}{T_0} \int_0^{\frac{T_0}{2}} A dt + \frac{1}{T_0} \int_{-\frac{T_0}{2}}^0 0 dt$$

$$C_0 = \frac{1}{T_0} \int_0^{\frac{T_0}{2}} A dt + \frac{1}{T_0} \int_{-\frac{T_0}{2}}^0 0 dt = \frac{1}{T_0} \int_0^{\frac{T_0}{2}} A dt$$

$$C_o = \frac{A}{T_0} \int_0^{\frac{T_0}{2}} dt = \frac{A}{T_0} [t] = \frac{A}{T_0} \left[\frac{T_0}{2} \right] = \frac{A}{2}$$

$$x(x) = C_0 + \sum_{n=1}^{\infty} C_n e^{j\omega t}$$

$$x(x) = \frac{A}{2} + \sum_{n=1}^{\infty} j \frac{A}{2\pi} [(-1)^2 - 1] e^{j\omega t}$$

(b) Determine the trigonometric Fourier series of x(t)

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi t}{T} + b_n \sin \frac{n\pi t}{T} \right)$$

$$a_0 = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) dt$$

$$a_0 = \frac{2}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) dt = \frac{2}{T_0} \int_0^{\frac{T_0}{2}} A dt + \frac{2}{T_0} \int_{-\frac{T_0}{2}}^0 0 dt$$

$$\omega = \frac{2\pi}{P} = \frac{2\pi}{T_0}$$

$$a_0 = \frac{2}{T_0} \int_0^{\frac{T_0}{2}} A dt + \frac{2}{T_0} \int_{-\frac{T_0}{2}}^0 0 dt = \frac{2}{T_0} \int_0^{\frac{T_0}{2}} A dt$$

$$a_0 = \frac{2A}{T_0} \int_0^{\frac{T_0}{2}} 1 dt = \frac{2A}{T_0} [t] = \frac{2A}{T_0} \left[\frac{T_0}{2} \right] = A$$

$$a_0 = A$$

$$avg = \frac{a_0}{2} = \frac{A}{2}$$

$$a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \cos(n\omega t) dt$$

$$a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \cos(n\omega t) dt = \frac{2}{T_0} \int_0^{\frac{T_0}{2}} A \cos(n\omega t) dt + \frac{2}{T_0} \int_{-\frac{T_0}{2}}^0 0 \cos(n\omega t) dt$$

$$\omega = \frac{2\pi}{P} = \frac{2\pi}{T_0}$$

$$a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \cos\left(\frac{2n\pi}{T_0} t\right) dt = \frac{2}{T_0} \int_0^{\frac{T_0}{2}} A \cos\left(\frac{2n\pi}{T_0} t\right) dt + \frac{2}{T_0} \int_{-\frac{T_0}{2}}^0 0 \cos\left(\frac{2n\pi}{T_0} t\right) dt$$

$$a_n = \frac{A}{T_0} \int_0^{\frac{T_0}{2}} \cos\left(\frac{2n\pi}{T_0} t\right) dt = \frac{A}{T_0} \left[\frac{\sin\left(\frac{2n\pi}{T_0} t\right)}{\frac{2n\pi}{T_0}} \right] = \frac{A}{\frac{2n\pi}{T_0} T_0} \left[\sin\left(\frac{2n\pi}{T_0} t\right) \right]$$

$$= \frac{A}{2n\pi} \left[\sin\left(\frac{2n\pi}{T_0} t\right) \right] = \frac{A}{2n\pi} \left[\sin\left(\frac{2n\pi}{T_0} t\right) - \sin\left(\frac{2n\pi}{T_0} t\right) \right] = \frac{A}{2n\pi} \left[\sin\left(\frac{2n\pi}{T_0} \frac{T_0}{2}\right) - \sin(0) \right]$$

$$= \frac{A}{2n\pi} \left[\sin\left(\frac{2n\pi}{T_0}\right) \frac{T_0}{2} \right] = \frac{A}{2n\pi} [\sin(n\pi)] = -\frac{A}{2n\pi} [0] = 0$$

$a_n = 0$

$$b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \sin(n\omega t) dt$$

$$b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \sin(n\omega t) dt = \frac{2}{T_0} \int_0^{\frac{T_0}{2}} A \sin(n\omega t) dt + \frac{2}{T_0} \int_{-\frac{T_0}{2}}^0 0 \sin(n\omega t) dt$$

$$\omega = \frac{2\pi}{P} = \frac{2\pi}{T_0}$$

$$b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \sin\left(\frac{2n\pi}{T_0}\right) t dt = \frac{2}{T_0} \int_0^{\frac{T_0}{2}} A \sin\left(\frac{2n\pi}{T_0}\right) t dt + \frac{2}{T_0} \int_{-\frac{T_0}{2}}^0 0 \sin\left(\frac{2n\pi}{T_0}\right) t dt$$

$$b_n = \frac{A}{T_0} \int_0^{\frac{T_0}{2}} \sin\left(\frac{2n\pi}{T_0}\right) t dt = -\frac{A}{T_0} \left[\frac{\cos\left(\frac{2n\pi}{T_0}\right) t}{\frac{2n\pi}{T_0}} \right] = -\frac{A}{\frac{2n\pi}{T_0} T_0} \left[\cos\left(\frac{2n\pi}{T_0}\right) t \right]$$

$$= -\frac{A}{2n\pi} \left[\cos\left(\frac{2n\pi}{T_0}\right) t \right] = -\frac{A}{2n\pi} \left[\cos\left(\frac{2n\pi}{T_0}\right) t - \cos\left(\frac{2n\pi}{T_0}\right) t \right]$$

$$= -\frac{A}{2n\pi} \left[\cos\left(\frac{2n\pi}{T_0}\right) \frac{T_0}{2} - \cos(0) \right]$$

$$= -\frac{A}{2n\pi} \left[\cos\left(\frac{2n\pi}{T_0}\right) \frac{T_0}{2} - 1 \right] = -\frac{A}{2n\pi} [\cos(n\pi) - 1] = -\frac{A}{2n\pi} [(-1)^n - 1]$$

$b_n = -\frac{A}{2n\pi} [(-1)^n - 1]$

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi t}{T} + b_n \sin \frac{n\pi t}{T} \right)$$

$$f(t) = A + \sum_{n=1}^{\infty} \left\{ a_n(0) - \frac{A}{2n\pi} [(-1)^n - 1] \sin \frac{n\pi t}{T} \right\}$$

$$f(t) = A + \sum_{n=1}^{\infty} \left\{ -\frac{A}{2n\pi} [(-1)^n - 1] \sin \frac{n\pi t}{T} \right\}$$

Verification

to convert the exponential Fourier Series back to trigonometric form.

$$\begin{cases} C_0 = \frac{a_0}{2} \\ C_n = \frac{1}{2}(a_n - jb_n) \\ C_{-n} = \frac{1}{2}(a_n + jb_n) \end{cases} \quad \begin{cases} a_n - jb_n = 2C_n \\ a_n + jb_n = 2C_{-n} \end{cases}$$

$$a_0 = A$$

$$C_0 = \frac{a_0}{2} = \frac{A}{2}$$

2.2.6 Fourier transform

1. Consider a periodic function defined by:

$$f(t) = \begin{cases} 2 & -3 \leq t < -2 \\ -1 & -2 \leq t < 2 \\ 2 & 2 \leq t < 3 \end{cases}$$

$$h(t+6) = h(t)$$

(a) Determine the Laplace transform of $h(t)$.

(b) Determine the average value of $h(t)$.

(c) Find the exponential Fourier series expansion of $h(t)$.

(d) Determine the Fourier transform of $h(t)$.

(e) Sketch the spectrum of $h(t)$ (Fourier transform of $h(t)$) for $n = \{\pm 2, \pm 1, 0\}$

(a) Determine the Laplace transform of $h(t)$.

$$\int_{-3}^{-2} 2e^{-st} dt + \int_{-2}^2 -1e^{-st} dt + \int_2^3 2e^{-st} dt$$

$$h(t) = \int \frac{e^{-st} f(t) dt}{1 - e^{-st}}$$

Laplace transform does not have a negative bounded, you need to shift it at $t > 0$

$$\begin{aligned} f(t) &= \int_0^2 -1e^{-st} dt + \int_2^4 2dt + \int_2^3 -1e^{-st} dt \\ &= -\int_0^2 e^{-st} dt + 2 \int_2^4 e^{-st} dt - \int_2^3 e^{-st} dt \\ &= -\left[\frac{e^{-st}}{-s}\right] + 2\left[\frac{e^{-st}}{-s}\right] - \left[\frac{e^{-st}}{-s}\right] \\ &= \left[\frac{e^{-2s} - e^0}{s}\right] - 2\left[\frac{e^{-4s} - e^{-2s}}{s}\right] + \left[\frac{e^{-3s} - e^{-2s}}{s}\right] \\ &= \frac{e^{-2s}}{s} - \frac{1}{s} - \frac{2e^{-4s}}{s} + \frac{2e^{-2s}}{s} + \frac{e^{-3s}}{s} - \frac{e^{-2s}}{s} \\ &= -\frac{1}{s} - \frac{2e^{-4s}}{s} + \frac{2e^{-2s}}{s} + \frac{e^{-3s}}{s} \\ f(t) &= \frac{e^{-3s} - 1 - 2e^{-4s} + 2e^{-2s}}{s} \end{aligned}$$

$$\begin{aligned} h(t) &= \frac{\int e^{-st} f(t) dt}{1 - e^{-st}} = \frac{\frac{e^{-3s} - 1 - 2e^{-4s} + 2e^{-2s}}{s}}{1 - e^{-st}} \\ &= \frac{e^{-3s} - 1 - 2e^{-4s} + 2e^{-2s}}{s} \times \frac{1}{1 - e^{-st}} = \frac{e^{-3s} - 1 - 2e^{-4s} + 2e^{-2s}}{s[1 - e^{-st}]} \\ h(t) &= \frac{e^{-3s} - 1 - 2e^{-4s} + 2e^{-2s}}{s - se^{-st}} \end{aligned}$$

(b) Determine the average value of $h(t)$.

$$avg = \frac{C_0}{2} = \frac{0}{2} = 0$$

$$C_0 = \frac{2}{T} \int_{-p}^p h(t) dt$$

$$C_0 = \frac{2}{6} \left[\int_{-3}^{-2} 2 dt - \int_{-2}^2 dt + \int_2^3 2 dt \right] = 0$$

(c) Find the exponential Fourier series expansion of h(t).

$$\begin{aligned} h(t) &= \frac{1}{6} \left[\int_{-3}^{-2} 2 e^{-jn\omega t} dt + \int_{-2}^2 -1 e^{-jn\omega t} dt + \int_2^3 2 e^{-jn\omega t} dt \right] \\ &= \frac{1}{6} \left[2 \int_{-3}^{-2} e^{-jn\omega t} dt - \int_{-2}^2 e^{-jn\omega t} dt + 2 \int_2^3 e^{-jn\omega t} dt \right] \\ &= \frac{1}{6} \left\{ 2 \left[\frac{e^{-jn\omega t}}{-jn\omega} \right] - \left[\frac{e^{-jn\omega t}}{-jn\omega} \right] + 2 \left[\frac{e^{-jn\omega t}}{-jn\omega} \right] \right\} \\ &= \frac{1}{6} \left\{ 2 \left[\frac{e^{j2n\omega} - e^{j3n\omega}}{-jn\omega} \right] - \left[\frac{e^{-j2n\omega} - e^{j2n\omega}}{-jn\omega} \right] + 2 \left[\frac{e^{-j3n\omega} - e^{-j2n\omega}}{-jn\omega} \right] \right\} \\ &= \frac{1}{6} \left\{ -2 \left[\frac{e^{j2n\omega} - e^{j3n\omega}}{jn\omega} \right] - \left[\frac{e^{j2n\omega} - e^{-j2n\omega}}{jn\omega} \right] - 2 \left[\frac{e^{-j3n\omega} - e^{-j2n\omega}}{jn\omega} \right] \right\} \\ &= \frac{1}{6} \left\{ -2 \left[\frac{e^{j2n\omega} - e^{j3n\omega}}{jn\omega} \right] - 2 \left[\frac{\sin(2n\omega)}{n\omega} \right] - 2 \left[\frac{e^{-j3n\omega} - e^{-j2n\omega}}{jn\omega} \right] \right\} \\ &= \frac{1}{6} \left\{ -2 \left[\frac{e^{j2n\omega} - e^{j3n\omega} + e^{-j3n\omega} - e^{-j2n\omega}}{jn\omega} \right] - 2 \left[\frac{\sin(2n\omega)}{n\omega} \right] \right\} \\ &= \frac{1}{6} \left\{ -2 \left[\frac{e^{-j3n\omega} - e^{j3n\omega} + e^{j2n\omega} - e^{-j2n\omega}}{jn\omega} \right] - 2 \left[\frac{\sin(2n\omega)}{n\omega} \right] \right\} \\ &= \frac{1}{6} \left\{ -2 \left[\frac{e^{-j3n\omega} - e^{j3n\omega}}{jn\omega} \right] - 2 \left[\frac{e^{j2n\omega} - e^{-j2n\omega}}{jn\omega} \right] - 2 \left[\frac{\sin(2n\omega)}{n\omega} \right] \right\} \\ &= \frac{1}{6} \left\{ 2 \left[\frac{e^{j3n\omega} - e^{-j3n\omega}}{jn\omega} \right] - 2 \left[\frac{e^{j2n\omega} - e^{-j2n\omega}}{jn\omega} \right] - 2 \left[\frac{\sin(2n\omega)}{n\omega} \right] \right\} \\ &= \frac{1}{6} \left\{ 2 \left[\frac{2\sin(3n\omega)}{n\omega} \right] - 2 \left[\frac{2\sin(2n\omega)}{n\omega} \right] - 2 \left[\frac{\sin(2n\omega)}{n\omega} \right] \right\} \\ &= \frac{2}{6} \left\{ 2 \left[\frac{\sin(3n\omega)}{n\omega} \right] - 2 \left[\frac{\sin(2n\omega)}{n\omega} \right] - \left[\frac{\sin(2n\omega)}{n\omega} \right] \right\} \\ &= \frac{1}{3} \left\{ 2 \left[\frac{\sin(3n\omega)}{n\omega} \right] - 2 \left[\frac{\sin(2n\omega)}{n\omega} \right] - \left[\frac{\sin(2n\omega)}{n\omega} \right] \right\} \\ &= \frac{1}{3} \left\{ 2 \left[\frac{\sin(3n\omega)}{n\omega} \right] \right\} - \frac{1}{3} \left\{ 2 \left[\frac{\sin(2n\omega)}{n\omega} \right] \right\} - \frac{1}{3} \left\{ \left[\frac{\sin(2n\omega)}{n\omega} \right] \right\} \\ &= \left\{ 2 \left[\frac{\sin(3n\omega)}{3n\omega} \right] \right\} - \frac{1}{3} \left\{ 2 \left[\frac{\sin(2n\omega)}{n\omega} \right] \right\} - \frac{1}{3} \left\{ \left[\frac{\sin(2n\omega)}{n\omega} \right] \right\} \\ &= \left\{ 2 \left[\frac{\sin(3n\omega)}{3n\omega} \right] \right\} - \frac{2}{3} \left\{ 2 \left[\frac{\sin(2n\omega)}{2n\omega} \right] \right\} - \frac{2}{3} \left\{ \left[\frac{\sin(2n\omega)}{2n\omega} \right] \right\} \\ &= \left\{ 2 \left[\frac{\sin(3n\omega)}{3n\omega} \right] \right\} - \frac{4}{3} \left\{ \left[\frac{\sin(2n\omega)}{2n\omega} \right] \right\} - \frac{2}{3} \left\{ \left[\frac{\sin(2n\omega)}{2n\omega} \right] \right\} \\ &= [2 \operatorname{sinc}(3n\omega)] - \frac{4}{3} [\operatorname{sinc}(2n\omega)] - \frac{2}{3} [\operatorname{sinc}(2n\omega)] \end{aligned}$$

$$\begin{aligned}
&= [2 \operatorname{sinc}(3n\omega)] + \left[-\frac{2}{3} - \frac{4}{3}\right] [\operatorname{sinc}(2n\omega)] \\
&= [2 \operatorname{sinc}(3n\omega)] + [-2] [\operatorname{sinc}(2n\omega)] \\
&= 2 \operatorname{sinc}(3n\omega) - 2 \operatorname{sinc}(2n\omega) \\
&= 2[\operatorname{sinc}(3n\omega) - \operatorname{sinc}(2n\omega)]
\end{aligned}$$

$$h(t) = 2 \operatorname{sinc}(3n\omega) - 2 \operatorname{sinc}(2n\omega)$$

(d) Determine the Fourier transform of $h(t)$.

$$\begin{aligned}
h(t) &= \int_{-3}^{-2} 2 e^{-j\omega t} dt + \int_{-2}^2 -1 e^{-j\omega t} dt + \int_2^3 2 e^{-j\omega t} dt \\
&= 2 \int_{-3}^{-2} e^{-j\omega t} dt - \int_{-2}^2 e^{-j\omega t} dt + 2 \int_2^3 e^{-j\omega t} dt \\
&= 2 \left[\frac{e^{-j\omega t}}{-j\omega} \right]_{-3}^{-2} - \left[\frac{e^{-j\omega t}}{-j\omega} \right]_{-2}^2 + 2 \left[\frac{e^{-j\omega t}}{-j\omega} \right]_2^3 \\
&= 2 \left[\frac{e^{j2\omega} - e^{j3\omega}}{-j\omega} \right] - \left[\frac{e^{-j2\omega} - e^{j2\omega}}{-j\omega} \right] + 2 \left[\frac{e^{-j3\omega} - e^{-j2\omega}}{-j\omega} \right] \\
&= -2 \left[\frac{e^{j2\omega} - e^{j3\omega}}{j\omega} \right] - \left[\frac{e^{j2\omega} - e^{-j2\omega}}{j\omega} \right] - 2 \left[\frac{e^{-j3\omega} - e^{-j2\omega}}{j\omega} \right] \\
&= -2 \left[\frac{e^{j2\omega} - e^{j3\omega} + e^{-j3\omega} - e^{-j2\omega}}{j\omega} \right] - \left[\frac{e^{j2\omega} - e^{-j2\omega}}{j\omega} \right] \\
&= -2 \left[\frac{e^{j2\omega} - e^{-j2\omega} - e^{j3\omega} + e^{-j3\omega}}{j\omega} \right] - \left[\frac{2 \sin(2\omega)}{\omega} \right] \\
&= -2 \left[\frac{e^{j2\omega} - e^{-j2\omega}}{j\omega} \right] - 2 \left[\frac{-e^{j3\omega} + e^{-j3\omega}}{j\omega} \right] - \left[\frac{2 \sin(2\omega)}{\omega} \right] \\
&= -2 \left[\frac{e^{j2\omega} - e^{-j2\omega}}{j\omega} \right] + 2 \left[\frac{e^{j3\omega} - e^{-j3\omega}}{j\omega} \right] - \left[\frac{2 \sin(2\omega)}{\omega} \right] \\
&= -2 \left[\frac{2 \sin(2\omega)}{\omega} \right] + 2 \left[\frac{2 \sin(3\omega)}{\omega} \right] - \left[\frac{2 \sin(2\omega)}{\omega} \right] \\
&= -4 \left[\frac{\sin(2\omega)}{\omega} \right] + 4 \left[\frac{\sin(3\omega)}{\omega} \right] - 2 \left[\frac{\sin(2\omega)}{\omega} \right] \\
&= 4 \left[\frac{\sin(3\omega)}{\omega} \right] - 6 \left[\frac{\sin(2\omega)}{\omega} \right] \\
&= 4[\operatorname{sinc}(3\omega)] - 6[\operatorname{sinc}(2\omega)] \\
&= 4 \operatorname{sinc}(3\omega) - 6 \operatorname{sinc}(2\omega)
\end{aligned}$$

$$h(t) = 4 \operatorname{sinc}(3\omega) - 6 \operatorname{sinc}(2\omega)$$

(e) Sketch the spectrum of $h(t)$ (Fourier transform of $h(t)$) for $n = \{\pm 2, \pm 1, 0\}$

$$h(t) = 4 \operatorname{sinc}(3\omega) - 6 \operatorname{sinc}(2\omega)$$

$$n = \{\pm 2, \pm 1, 0\}$$

$$\omega = \frac{2\pi}{6} = \frac{\pi}{3}$$

$$h(t) = 4 \operatorname{sinc}\left(3 \frac{\pi}{3}\right) - 6 \operatorname{sinc}\left(2 \frac{\pi}{3}\right) = 4 \operatorname{sinc}(\pi) - 6 \operatorname{sinc}\left(\frac{2}{3}\pi\right)$$

$$h(t) = 4 \operatorname{sinc}(3n\omega) - 6 \operatorname{sinc}(2n\omega) = 4 \operatorname{sinc}(n\pi) - 6 \operatorname{sinc}\left(\frac{2}{3}n\pi\right)$$

$$h(t) = 4 \operatorname{sinc}[(n)\pi] - 6 \operatorname{sinc}\left[(n) \frac{2}{3}\pi\right]$$

$$n = \{\pm 2, \pm 1, 0\}$$

$$F\{h(t)\} = 4 \operatorname{sinc}[(-2)\pi] - 6 \operatorname{sinc}\left[(-2) \frac{2}{3}\pi\right] + 4 \operatorname{sinc}[(2)\pi] - 6 \operatorname{sinc}\left[(2) \frac{2}{3}\pi\right]$$

$$+ 4 \operatorname{sinc}[(-1)\pi]$$

$$- 6 \operatorname{sinc}\left[(-1) \frac{2}{3}\pi\right] + 4 \operatorname{sinc}[(1)\pi] - 6 \operatorname{sinc}\left[(1) \frac{2}{3}\pi\right] + 4 \operatorname{sinc}[(0)\pi] - 6 \operatorname{sinc}\left[(0) \frac{2}{3}\pi\right]$$

$$F\{h(t)\} = 4 \operatorname{sinc}[-2\pi] - 6 \operatorname{sinc}\left[-\frac{4}{3}\pi\right] + 4 \operatorname{sinc}[2\pi] - 6 \operatorname{sinc}\left[\frac{4}{3}\pi\right] + 4 \operatorname{sinc}[-\pi]$$

$$- 6 \operatorname{sinc}\left[-\frac{2}{3}\pi\right] + 4 \operatorname{sinc}[\pi] - 6 \operatorname{sinc}\left[\frac{2}{3}\pi\right] + 4 \operatorname{sinc}(0) - 6 \operatorname{sinc}(0)$$

$$F\{h(t)\} = 0 - 6 \operatorname{sinc}\left[-\frac{4}{3}\pi\right] + 0 - 0 - 0 - 6 \operatorname{sinc}\left[-\frac{2}{3}\pi\right] + 0 - 6 \operatorname{sinc}\left[\frac{2}{3}\pi\right] + 0 - 0$$

$$F\{h(t)\} = -6 \operatorname{sinc}\left[-\frac{4}{3}\pi\right] - 6 \operatorname{sinc}\left[-\frac{2}{3}\pi\right] - 6 \operatorname{sinc}\left[\frac{2}{3}\pi\right]$$

$$F\{h(t)\} = -6 \frac{\sin\left[-\frac{4}{3}\pi\right]}{-\frac{4}{3}\pi} - 6 \frac{\sin\left[-\frac{2}{3}\pi\right]}{-\frac{2}{3}\pi} - 6 \frac{\sin\left[\frac{2}{3}\pi\right]}{\frac{2}{3}\pi}$$

$$F\{h(t)\} = 6 \frac{\sin\left[-\frac{4}{3}\pi\right]}{\frac{4}{3}\pi} + 6 \frac{\sin\left[-\frac{2}{3}\pi\right]}{\frac{2}{3}\pi} - 6 \frac{\sin\left[\frac{2}{3}\pi\right]}{\frac{2}{3}\pi}$$

$$F\{h(t)\} = 6 \frac{\frac{\sqrt{3}}{2}}{\frac{4}{3}\pi} - 6 \frac{\frac{\sqrt{3}}{2}}{\frac{2}{3}\pi} - 6 \frac{\frac{\sqrt{3}}{2}}{\frac{2}{3}\pi} = 6 \frac{\frac{\sqrt{3}}{2}}{\frac{4}{3}\pi} - 12 \frac{\frac{\sqrt{3}}{2}}{\frac{2}{3}\pi} = 6 \frac{\frac{\sqrt{3}}{2}}{\frac{4\pi}{3}} - 12 \frac{\frac{\sqrt{3}}{2}}{\frac{2\pi}{3}}$$

$$= 6 \frac{\sqrt{3}}{2} \times \frac{3}{4\pi} - 12 \frac{\sqrt{3}}{2} \times \frac{3}{2\pi}$$

$$= 6 \frac{3\sqrt{3}}{8\pi} - 12 \frac{3\sqrt{3}}{4\pi} = \frac{18\sqrt{3}}{8\pi} - \frac{36\sqrt{3}}{4\pi} = \frac{9\sqrt{3}}{4\pi} - \frac{9\sqrt{3}}{\pi} = \frac{9\sqrt{3} - 36\sqrt{3}}{4\pi} = -\frac{27\sqrt{3}}{4\pi}$$

$$F\{h(t)\} = -\frac{27\sqrt{3}}{4\pi} = -3.721470044$$

2. Consider a continuous – time LTI (Linear time invariant) system described by:

$$2 \frac{dy(t)}{dx} - y(t) = \frac{d^2x(t)}{dt^2} - \frac{dx(t)}{dx} - 2x(t)$$

Using the Fourier transform, find the output $y(t)$ if the input $x(t) = e^{-t} u(t)$

$$\begin{aligned} 2 \frac{dy(t)}{dx} - y(t) &= \frac{d^2x(t)}{dt^2} - \frac{dx(t)}{dx} - 2x(t) \\ \mathcal{F}\left\{2 \frac{dy(t)}{dx} - y(t)\right\} &= \mathcal{F}\left\{\frac{d^2x(t)}{dt^2} - \frac{dx(t)}{dx} - 2x(t)\right\} \\ \{2(j\omega)y(\omega) - y(\omega)\} &= \{(j\omega)^2x(\omega) - (j\omega)x(\omega) - 2x(\omega)\} \\ y(\omega)[2(j\omega) - 1] &= x(\omega)[(j\omega)^2 - (j\omega) - 2] \\ y(\omega) &= \frac{[(j\omega)^2 - (j\omega) - 2]}{[2(j\omega) - 1]}x(\omega) \end{aligned}$$

$$\begin{aligned} x(t) &= e^{-t} u(t) = e^{-at} u(t) \rightarrow a = 1 \\ x(\omega) &= \mathcal{F}\{x(t)\} = \mathcal{F}\{e^{-at} u(t)\} = \frac{1}{a + j\omega} \rightarrow \frac{1}{1 + j\omega} \end{aligned}$$

$$\begin{aligned} y(\omega) &= \frac{(j\omega)^2 - (j\omega) - 2}{2(j\omega) - 1} \times \frac{1}{1 + j\omega} \\ y(\omega) &= \frac{s^2 - s - 2}{2s - 1} \times \frac{1}{1 + s} = \frac{s^2 - s - 2}{[2s - 1][s + 1]} = \frac{[s - 2][s + 1]}{[2s - 1][s + 1]} = \frac{[s - 2]}{[2s - 1]} \\ y(\omega) &= \frac{s}{[2s - 1]} - \frac{2}{[2s - 1]} \end{aligned}$$

$$\begin{aligned} y(\omega) &= \frac{s - 2}{[2s - 1]} = \frac{2}{2} \frac{s - 2}{[2s - 1]} = \frac{2s - 2^2}{2[2s - 1]} = \frac{2s - 1 - 3}{2[2s - 1]} = \frac{2s - 1}{2[2s - 1]} - \frac{3}{2[2s - 1]} \\ &= \frac{1}{2} - \frac{3}{2[2s - 1]} \\ &= \frac{1}{2} - \frac{3}{4 \left[\frac{2s - 1}{2}\right]} = \frac{1}{2} - \frac{3}{4 \left[s - \frac{1}{2}\right]} = \frac{1}{2} - \frac{3}{4 \left[s - 0.5\right]} \\ y(\omega) &= \frac{1}{2} - \frac{3}{4[s - 0.5]} \end{aligned}$$

$$\begin{aligned} y(t) &= \mathcal{F}^{-1}\{y(\omega)\} = \mathcal{F}^{-1}\left\{\frac{1}{2} - \frac{3}{4[s - 0.5]}\right\} = \mathcal{F}^{-1}\left\{\frac{1}{2}\right\} - \mathcal{F}^{-1}\left\{\frac{3}{4[s - 0.5]}\right\} \\ y(t) &= \frac{1}{2} \mathcal{F}^{-1}\{1\} - \frac{3}{4} \mathcal{F}^{-1}\left\{\frac{1}{[s - 0.5]}\right\} = \frac{1}{2} \delta(t) - \frac{3}{4} e^{0.5t} u(-t) \end{aligned}$$

$$y(t) = \frac{1}{2} \delta(t) - \frac{3}{4} e^{0.5t} u(-t)$$

3. Calculate the Fourier transform of the windowed cosine function

$$f(t) = \cos(\omega_o t) \left[u\left(t + \frac{T}{2}\right) - u\left(t - \frac{T}{2}\right) \right]$$

$$\cos(\omega_o t), \quad -\frac{T}{2} < t < \frac{T}{2}$$

$$\begin{aligned} \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(\omega_o t) e^{-j\omega t} dt &= \int_{-\frac{T}{2}}^{\frac{T}{2}} \left[\frac{e^{j\omega_o t} + e^{-j\omega_o t}}{2} \right] e^{-j\omega t} dt = \frac{1}{2} \int_{-\frac{T}{2}}^{\frac{T}{2}} [e^{j\omega_o t} + e^{-j\omega_o t}] e^{-j\omega t} dt \\ &= \frac{1}{2} \int_{-\frac{T}{2}}^{\frac{T}{2}} [e^{j\omega_o t} e^{-j\omega t} + e^{-j\omega_o t} e^{-j\omega t}] dt = \frac{1}{2} \int_{-\frac{T}{2}}^{\frac{T}{2}} [e^{j\omega_o t - j\omega t} + e^{-j\omega_o t - j\omega t}] dt \\ &= \frac{1}{2} \int_{-\frac{T}{2}}^{\frac{T}{2}} [e^{j(\omega_o - \omega)t} + e^{-j(\omega_o + \omega)t}] dt = \frac{1}{2} \int_{-\frac{T}{2}}^{\frac{T}{2}} [e^{j(\omega_o - \omega)t} + e^{-j(\omega_o + \omega)t}] dt \\ &= \frac{1}{2} \left[\frac{e^{j(\omega_o - \omega)t}}{j(\omega_o - \omega)} + \frac{e^{-j(\omega_o + \omega)t}}{-j(\omega_o + \omega)} \right] = \frac{1}{2} \left[\frac{e^{j(\omega_o - \omega)t}}{j(\omega_o - \omega)} - \frac{e^{-j(\omega_o + \omega)t}}{j(\omega_o + \omega)} \right] \\ &= \frac{1}{2} \left[\frac{e^{j(\omega_o - \omega)\frac{T}{2}} - e^{-j(\omega_o - \omega)\frac{T}{2}}}{j(\omega_o - \omega)} - \frac{e^{-j(\omega_o + \omega)\frac{T}{2}} - e^{j(\omega_o + \omega)\frac{T}{2}}}{j(\omega_o + \omega)} \right] \\ &= \frac{1}{2} \left[\frac{e^{j(\omega_o - \omega)\frac{T}{2}} - e^{-j(\omega_o - \omega)\frac{T}{2}}}{j(\omega_o - \omega)} + \frac{e^{j(\omega_o + \omega)\frac{T}{2}} - e^{-j(\omega_o + \omega)\frac{T}{2}}}{j(\omega_o + \omega)} \right] \\ &= \left[\frac{\sin(\omega_o - \omega)\frac{T}{2}}{(\omega_o - \omega)} + \frac{\sin(\omega_o + \omega)\frac{T}{2}}{(\omega_o + \omega)} \right] = \frac{\sin(\omega_o - \omega)\frac{T}{2}}{(\omega_o - \omega)} + \frac{\sin(\omega_o + \omega)\frac{T}{2}}{(\omega_o + \omega)} \\ &= \frac{\frac{T}{2} \sin(\omega_o - \omega)\frac{T}{2}}{\frac{T}{2}(\omega_o - \omega)} + \frac{\frac{T}{2} \sin(\omega_o + \omega)\frac{T}{2}}{\frac{T}{2}(\omega_o + \omega)} = \frac{T \sin(\omega_o - \omega)\frac{T}{2}}{2(\omega_o - \omega)\frac{T}{2}} + \frac{T \sin(\omega_o + \omega)\frac{T}{2}}{2(\omega_o + \omega)\frac{T}{2}} \\ &= \frac{T}{2} \text{sinc}(\omega_o - \omega)\frac{T}{2} + \frac{T}{2} \text{sinc}(\omega_o + \omega)\frac{T}{2} \end{aligned}$$

$$C_n = \frac{T}{2} \left[\text{sinc}(\omega_o - \omega)\frac{T}{2} + \text{sinc}(\omega_o + \omega)\frac{T}{2} \right]$$

4. Find the Fourier transform of the shifted form of the windowed cosine function

$$g(t) = \cos(\omega_o t) [u(t) - u(t - T)]$$

$$\begin{aligned}
 g(t) &= \cos(\omega_o t) [u(t - 0) - u(t - T)] \\
 g(t) &= \cos(\omega_o t), \quad 0 < t < T \\
 \int_0^T \cos(\omega_o t) e^{-j\omega t} dt &= \int_0^T \left[\frac{e^{j\omega_o t} + e^{-j\omega_o t}}{2} \right] e^{-j\omega t} dt = \frac{1}{2} \int_0^T [e^{j\omega_o t} + e^{-j\omega_o t}] e^{-j\omega t} dt \\
 &= \frac{1}{2} \int_0^T [e^{j\omega_o t} e^{-j\omega t} + e^{-j\omega_o t} e^{-j\omega t}] dt = \frac{1}{2} \int_0^T [e^{j\omega_o t - j\omega t} + e^{-j\omega_o t - j\omega t}] dt \\
 &= \frac{1}{2} \int_0^T [e^{j(\omega_o - \omega)t} + e^{-j(\omega_o + \omega)t}] dt = \frac{1}{2} \left[\frac{e^{j(\omega_o - \omega)t}}{j(\omega_o - \omega)} + \frac{e^{-j(\omega_o + \omega)t}}{-j(\omega_o + \omega)} \right] \\
 &= \frac{1}{2} \left[\frac{e^{j(\omega_o - \omega)T}}{j(\omega_o - \omega)} - \frac{e^{-j(\omega_o + \omega)T}}{j(\omega_o + \omega)} \right] = \frac{1}{2} \left[\frac{e^{j(\omega_o - \omega)T} - e^{j(\omega_o - \omega)0}}{j(\omega_o - \omega)} - \frac{e^{-j(\omega_o + \omega)T} - e^{-j(\omega_o + \omega)0}}{j(\omega_o + \omega)} \right] \\
 &= \frac{1}{2} \left[\frac{e^{j(\omega_o - \omega)T} - e^0}{j(\omega_o - \omega)} - \frac{e^{-j(\omega_o + \omega)T} - e^0}{j(\omega_o + \omega)} \right] = \frac{1}{2} \left[\frac{e^{j(\omega_o - \omega)T} - 1}{j(\omega_o - \omega)} - \frac{e^{-j(\omega_o + \omega)T} - 1}{j(\omega_o + \omega)} \right] \\
 &= \frac{1}{2} \left[\frac{j \sin(\omega_o - \omega)T + \cos(\omega_o - \omega)T - 1}{j(\omega_o - \omega)} - \frac{\cos(\omega_o + \omega)T - j \cos(\omega_o + \omega)T - 1}{j(\omega_o + \omega)} \right] \\
 &= \frac{1}{2} \left[\frac{j \sin(\omega_o - \omega)T + \cos(\omega_o - \omega)T - 1}{j(\omega_o - \omega)} \right] - \frac{1}{2} \left[\frac{\cos(\omega_o + \omega)T - j \cos(\omega_o + \omega)T - 1}{j(\omega_o + \omega)} \right] \\
 &= \frac{1}{2j} \left[\frac{j \sin(\omega_o - \omega)T + \cos(\omega_o - \omega)T - 1}{j(\omega_o - \omega)} \right] - \frac{1}{2j} \left[\frac{\cos(\omega_o + \omega)T - j \sin(\omega_o + \omega)T - 1}{j(\omega_o + \omega)} \right] \\
 &= \frac{1}{2} \left[\frac{-\sin(\omega_o - \omega)T + j \cos(\omega_o - \omega)T - j}{-(\omega_o - \omega)} \right] - \frac{1}{2} \left[\frac{j \cos(\omega_o + \omega)T + \sin(\omega_o + \omega)T - j}{-(\omega_o + \omega)} \right] \\
 &= \frac{1}{2} \left[\frac{\sin(\omega_o - \omega)T}{(\omega_o - \omega)} - \frac{j \cos(\omega_o - \omega)T}{(\omega_o - \omega)} + \frac{j}{(\omega_o - \omega)} \right] \\
 &\quad - \frac{1}{2} \left[-\frac{\sin(\omega_o + \omega)T}{(\omega_o + \omega)} - \frac{j \cos(\omega_o + \omega)T}{(\omega_o + \omega)} + \frac{j}{(\omega_o + \omega)} \right]
 \end{aligned}$$

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$\cos^2 \theta = 1 - \sin^2 \theta$$

$$\cos \theta = \sqrt{1 - \sin^2 \theta}$$

$$\sqrt{\frac{y}{x^2}} = \frac{\sqrt{y}}{\sqrt{x^2}} = \frac{\sqrt{y}}{x}$$

$$\begin{aligned}
 C_n &= \frac{1}{2} \left[T \operatorname{sinc}(\omega_o - \omega)T - \frac{j \sqrt{1 - \sin^2(\omega_o - \omega)T}}{(\omega_o - \omega)} + \frac{j}{(\omega_o - \omega)} \right] \\
 &\quad - \frac{1}{2} \left[-T \operatorname{sinc}(\omega_o + \omega)T - \frac{j \sqrt{1 - \sin^2(\omega_o + \omega)T}}{(\omega_o + \omega)} + \frac{j}{(\omega_o + \omega)} \right] \\
 C_n &= \frac{1}{2} \left[T \operatorname{sinc}(\omega_o - \omega)T - j \sqrt{\frac{1 - \sin^2(\omega_o - \omega)T}{(\omega_o - \omega)^2}} + \frac{j}{(\omega_o - \omega)} \right]
 \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2} \left[-Tsinc(\omega_o + \omega)T - j \sqrt{\frac{1 - \sin^2(\omega_o + \omega)T}{(\omega_o + \omega)^2}} + \frac{j}{(\omega_o + \omega)} \right] \\
C_n &= \frac{1}{2} \left[Tsinc(\omega_o - \omega)T - j \sqrt{\frac{1}{(\omega_o - \omega)^2} - \frac{\sin^2(\omega_o - \omega)T}{(\omega_o - \omega)^2}} + \frac{j}{(\omega_o - \omega)} \right] \\
& -\frac{1}{2} \left[-Tsinc(\omega_o + \omega)T - j \sqrt{\frac{1}{(\omega_o + \omega)^2} - \frac{\sin^2(\omega_o + \omega)T}{(\omega_o + \omega)^2}} + \frac{j}{(\omega_o + \omega)} \right] \\
C_n &= \frac{1}{2} \left[Tsinc(\omega_o - \omega)T - j \sqrt{\frac{1}{(\omega_o - \omega)^2} - \text{sinc}^2(\omega_o - \omega)T} + \frac{j}{(\omega_o - \omega)} \right] \\
& -\frac{1}{2} \left[-Tsinc(\omega_o + \omega)T - j \sqrt{\frac{1}{(\omega_o + \omega)^2} - \text{sinc}^2(\omega_o + \omega)T} + \frac{j}{(\omega_o + \omega)} \right] \\
C_n &= \frac{1}{2} \left[T\text{sinc}^2(\omega_o - \omega)T + \frac{1}{(\omega_o - \omega)^2} - \text{sinc}^2(\omega_o - \omega)T - \frac{1}{(\omega_o - \omega)} \right] \\
& -\frac{1}{2} \left[-T\text{sinc}^2(\omega_o + \omega)T + \frac{1}{(\omega_o + \omega)^2} - \text{sinc}^2(\omega_o + \omega)T - \frac{1}{(\omega_o + \omega)^2} \right] \\
C_n &= \frac{T}{2} \text{sinc}^2(\omega_o - \omega)T + \frac{1}{2} \frac{1}{(\omega_o - \omega)^2} - \frac{1}{2} \text{sinc}^2(\omega_o - \omega)T - \frac{1}{2} \frac{1}{(\omega_o - \omega)} \\
& + \frac{T}{2} \text{sinc}^2(\omega_o + \omega)T - \frac{1}{2} \frac{1}{(\omega_o + \omega)^2} + \frac{1}{2} \text{sinc}^2(\omega_o + \omega)T + \frac{1}{2} \frac{1}{(\omega_o + \omega)^2} \\
C_n &= \frac{T}{2} \text{sinc}^2(\omega_o - \omega)T + \frac{1}{2} \frac{1}{(\omega_o - \omega)^2} - \frac{1}{2} \text{sinc}^2(\omega_o - \omega)T - \frac{1}{2} \frac{1}{(\omega_o - \omega)} \\
& + \frac{T}{2} \text{sinc}^2(\omega_o + \omega)T - \frac{1}{2} \frac{1}{(\omega_o + \omega)^2} + \frac{1}{2} \text{sinc}^2(\omega_o + \omega)T + \frac{1}{2} \frac{1}{(\omega_o + \omega)^2} \\
C_n &= \frac{T}{2} \text{sinc}^2(\omega_o - \omega)T - \frac{1}{2} \text{sinc}^2(\omega_o - \omega)T + \frac{T}{2} \text{sinc}^2(\omega_o + \omega)T + \frac{1}{2} \text{sinc}^2(\omega_o + \omega)T \\
C_n &= \frac{T}{2} \text{sinc}^2(\omega_o - \omega)T + \frac{T}{2} \text{sinc}^2(\omega_o + \omega)T - \frac{1}{2} \text{sinc}^2(\omega_o - \omega)T + \frac{1}{2} \text{sinc}^2(\omega_o + \omega)T \\
C_n &= \frac{T}{2} [\text{sinc}^2(\omega_o - \omega)T + \text{sinc}^2(\omega_o + \omega)T] + \frac{1}{2} [\text{sinc}^2(\omega_o + \omega)T - \text{sinc}^2(\omega_o - \omega)T] \\
C_n &= \frac{T}{2} [\text{sinc}^2(\omega_o - \omega)T + \text{sinc}^2(\omega_o + \omega)T] + \frac{1}{2} [\text{sinc}^2(\omega_o + \omega)T - \text{sinc}^2(\omega_o - \omega)T]
\end{aligned}$$

5. Calculate the Fourier transform of the windowed sine function

$$f(t) = \sin(2t) [u(t+1) - u(t-1)]$$

$$\begin{aligned}
 f(t) &= \sin(2t), \quad -1 < t < 1 \\
 \int_{-1}^1 \sin(2t) e^{-j\omega t} dt &= \int_{-1}^1 \left[\frac{e^{j2t} - e^{-j2t}}{2j} \right] e^{-j\omega t} dt = \frac{1}{2j} \int_{-1}^1 [e^{j2t} - e^{-j2t}] e^{-j\omega t} dt \\
 &= \frac{1}{2j} \int_{-1}^1 [e^{-j\omega t} e^{j2t} - e^{-j\omega t} e^{-j2t}] dt = \frac{1}{2j} \int_{-1}^1 [e^{-j\omega t + j2t} - e^{-j\omega t - j2t}] dt \\
 &= \frac{1}{2j} \int_{-1}^1 [e^{(-j\omega + j2)t} - e^{(-j\omega - j2)t}] dt = \frac{1}{2j} \int_{-1}^1 [e^{j(-\omega + 2)t} - e^{-j(\omega + 2)t}] dt \\
 &= \frac{1}{2j} \left[\frac{e^{j(-\omega + 2)t}}{j(-\omega + 2)} - \frac{e^{-j(\omega + 2)t}}{-j(\omega + 2)} \right] = \frac{1}{2j} \left[\frac{e^{j(-\omega + 2)t}}{j(2 - \omega)} + \frac{e^{-j(\omega + 2)t}}{j(\omega + 2)} \right] \\
 &= \frac{1}{2j} \left[\frac{e^{j(2 - \omega)} - e^{-j(2 - \omega)}}{j(2 - \omega)} + \frac{e^{-j(\omega + 2)} - e^{j(\omega + 2)}}{j(\omega + 2)} \right] \\
 &= \frac{1}{2j} \left[\frac{e^{j(2 - \omega)} - e^{-j(2 - \omega)}}{j(2 - \omega)} - \frac{e^{j(\omega + 2)} - e^{-j(\omega + 2)}}{j(\omega + 2)} \right] \\
 &= \frac{1}{j} \left[\frac{e^{j(2 - \omega)} - e^{-j(2 - \omega)}}{2(2 - \omega)} - \frac{e^{j(\omega + 2)} - e^{-j(\omega + 2)}}{2(\omega + 2)} \right] = \frac{1}{j} \left[\frac{\sin(2 - \omega)}{(2 - \omega)} - \frac{\sin(\omega + 2)}{(\omega + 2)} \right] \\
 &= \frac{1}{j} [\text{sinc}(2 - \omega) - \text{sinc}(\omega + 2)]
 \end{aligned}$$

$$C_n = -j[\text{sinc}(2 - \omega) - \text{sinc}(\omega + 2)]$$

2.2.7 Z – transform

1. Using Z

– transform, solve the following difference equation with the given initial conditions:

$$y[k] - 3y[k - 1] = x[k] \quad \text{with } x[k] = 4u[k], \quad y[-1] = 1.$$

$$y[k] - 3y[k - 1] = x[k]$$

$$\mathcal{Z}\{y[k] - 3y[k - 1]\} = \mathcal{Z}\{x[k]\}$$

Z transform Table:

$$y[k] = y[z]$$

$$y[k - 1] = \{z^{-1}y[z] + y[-1]\} = \{z^{-1}y[z] + 1\}$$

$$x[k] = 4u[k] = 4\left[\frac{z}{z - 1}\right]$$

$$\mathcal{Z}\{y[k]\} - 3\mathcal{Z}\{y[k - 1]\} = \mathcal{Z}\{4u[k]\}$$

$$\mathcal{Z}\{y[k]\} - 3\mathcal{Z}\{y[k - 1]\} = 4\mathcal{Z}\{u[k]\}$$

$$y[z] - 3\{z^{-1}y[z] + y[-1]\} = 4\left[\frac{z}{z - 1}\right]$$

$$y[z] - 3\{z^{-1}y[z] + 1\} = 4\left[\frac{z}{z - 1}\right]$$

$$y[z] - 3\{z^{-1}y[z]\} - 3 = \left[\frac{4z}{z - 1}\right]$$

$$y[z] - 3\{z^{-1}y[z]\} = 3 + \left[\frac{4z}{z - 1}\right]$$

$$y[z]\{1 - 3[z^{-1}]\} = 3 + \left[\frac{4z}{z - 1}\right]$$

$$y[z] = \frac{3 + \left[\frac{4z}{z - 1}\right]}{\{1 - 3[z^{-1}]\}}$$

$$y[z] = \frac{3}{\{1 - 3[z^{-1}]\}} + \frac{\left[\frac{4z}{z - 1}\right]}{\{1 - 3[z^{-1}]\}}$$

$$y[z] = \frac{3}{\left[1 - \frac{3}{z}\right]} + \frac{\left[\frac{4z}{z - 1}\right]}{\left[1 - \frac{3}{z}\right]}$$

$$y[z] = \frac{3}{\left[\frac{z - 3}{z}\right]} + \frac{\left[\frac{4z}{z - 1}\right]}{\left[\frac{z - 3}{z}\right]}$$

$$y[z] = \frac{3}{\frac{[z - 3]}{[z]}} + \frac{\frac{[4z]}{[z - 1]}}{\frac{[z - 3]}{[z]}}$$

$$y[z] = \frac{3[z]}{[z - 3]} + \frac{[4z][z]}{[z - 1][z - 3]}$$

$$y[z] = \frac{3[z]}{[z - 3]} + \frac{[4z^2]}{[z - 1][z - 3]}$$

$$y[z] = \frac{3z}{[z - 3]} + \frac{4z^2}{[z - 1][z - 3]}$$

partial fraction:

$$y[z] = \frac{4z^2}{[z-1][z-3]}$$

$$\frac{y[z]}{z} = \frac{4z}{[z-1][z-3]} = \frac{A}{[z-1]} + \frac{B}{[z-3]}$$

$$\frac{4z[z-1][z-3]}{[z-1][z-3]} = \frac{A[z-1][z-3]}{[z-1]} + \frac{B[z-1][z-3]}{[z-3]}$$

$$4z = A[z-3] + B[z-1]$$

when $z = 1$ we get $4(1) = A[(1) - 3] + B[(1) - 1]$

$$4 = A[1 - 3] + B[1 - 1]$$

$$4 = A[-2] + B[0]$$

$$4 = A[-2]$$

$$A = \frac{4}{-2} = -2$$

$$4z = A[z-3] + B[z-1]$$

when $z = 3$ we get $4(3) = A[(3) - 3] + B[(3) - 1]$

$$12 = A[3 - 3] + B[3 - 1]$$

$$12 = A[0] + B[2]$$

$$12 = B[2]$$

$$B = \frac{12}{2} = 6$$

$$\frac{y[z]}{z} = \frac{4z}{[z-1][z-3]} = \frac{A}{[z-1]} + \frac{B}{[z-3]}$$

$$\frac{y[z]}{z} = \frac{4z}{[z-1][z-3]} = \frac{-2}{[z-1]} + \frac{6}{[z-3]} = -\frac{2}{[z-1]} + \frac{6}{[z-3]}$$

$$y[z] = \frac{4z^2}{[z-1][z-3]} = \frac{Az}{[z-1]} + \frac{Bz}{[z-3]} = -\frac{2z}{[z-1]} + \frac{6z}{[z-3]}$$

the inverse of the Z transform: $z^{-1}y[z] = y[k]$

$$y[z] = \frac{3z}{[z-3]} + \frac{4z^2}{[z-1][z-3]} = \frac{3z}{[z-3]} - \frac{2z}{[z-1]} + \frac{6z}{[z-3]}$$

$$y[k] = z^{-1}y[z] = z^{-1} \left\{ \frac{3z}{[z-3]} - \frac{2z}{[z-1]} + \frac{6z}{[z-3]} \right\}$$

$$y[k] = z^{-1}y[z] = z^{-1} \left\{ \frac{3z}{[z-3]} \right\} - z^{-1} \left\{ \frac{2z}{[z-1]} \right\} + z^{-1} \left\{ \frac{6z}{[z-3]} \right\}$$

Z transform Table:

$$\delta[k] = 1 \quad u[k] = \frac{z}{z-1} \quad b^k = \frac{z}{z-b} \quad h_k = \frac{z}{z-1} \quad k = \frac{1}{(z-1)^2}$$

$$kb^{k-1} = \frac{z}{(z-b)^2} \quad (k^2 - k) = \frac{z}{(z-1)^3} \quad k^2 = \frac{z(z+1)}{(z-1)^3}$$

$$y[k] = z^{-1}y[z] = z^{-1} \left\{ \frac{3z}{[z-3]} \right\} - z^{-1} \left\{ \frac{2z}{[z-1]} \right\} + z^{-1} \left\{ \frac{6z}{[z-3]} \right\}$$

$$y[k] = z^{-1}y[z] = 3z^{-1} \left\{ \frac{z}{[z-3]} \right\} - 2z^{-1} \left\{ \frac{z}{[z-1]} \right\} + 6z^{-1} \left\{ \frac{z}{[z-3]} \right\}$$

$$\begin{aligned}
y[k] &= z^{-1}y[z] = 3(3)^k - 2(h_k) + 6(3)^k \\
y[k] &= 3(3)^k - 2(h_k) + 6(3)^k \\
y[k] &= [3 + 6](3)^k - 2(h_k)
\end{aligned}$$

$$y[k] = 9(3)^k - 2(h_k)$$

2. Determine $x[4]$ given

$$\begin{aligned}
2x[k + 2] - x[k + 1] + x[k] &= -k^2 & x[0] &= 1 & x[1] &= 3 \\
2x[k + 2] - x[k + 1] + x[k] &= -k^2 \\
Z\{2x[k + 2] - x[k + 1] + x[k]\} &= Z\{-k^2\}
\end{aligned}$$

Z transform Table:

$$\begin{aligned}
x[k] &= x[z] \\
x[k + 1] &= \{Z y[z] - Zx[0]\} = \{Z x[z] - Z[1]\} = \{Z x[z] - Z\} \\
x[k + 2] &= \{Z^2 x[z] - Z^2 x[0] - Zx[1]\} = \{Z^2 x[z] - Z^2 [1] - Z[3]\} \\
&= \{Z^2 x[z] - Z^2 - 3Z\} \\
k^2 &= \frac{z(z + 1)}{(z - 1)^3}
\end{aligned}$$

$$\begin{aligned}
Z\{2x[k + 2] - x[k + 1] + x[k]\} &= Z\{-k^2\} \\
2Z\{x[k + 2]\} - Z\{x[k + 1]\} + Z\{x[k]\} &= -Z\{k^2\} \\
2\{Z^2 x[z] - Z^2 - 3Z\} - \{Z x[z] - Z\} + x[z] &= -\left[\frac{z(z + 1)}{(z - 1)^3}\right] \\
2\{Z^2 x[z]\} - 2\{Z^2\} - 2\{3Z\} - \{Z x[z]\} - \{-Z\} + x[z] &= -\frac{(z^2 + z)}{(z - 1)^3} \\
2Z^2 x[z] - 2Z^2 - 6Z - Z x[z] + Z + x[z] &= -\frac{(z^2 + z)}{(z - 1)^3} \\
2Z^2 x[z] - Z x[z] + x[z] &= 2Z^2 + 6Z - Z - \frac{(z^2 + z)}{(z - 1)^3} \\
x[z][2Z^2 - Z + 1] &= 2Z^2 + 5Z - \frac{(z^2 + z)}{(z - 1)^3} \\
x[z] &= \frac{2Z^2 + 5Z - \frac{(z^2 + z)}{(z - 1)^3}}{[2Z^2 - Z + 1]} = \frac{2Z^2 + 5Z}{[2Z^2 - Z + 1]} - \frac{\frac{(z^2 + z)}{(z - 1)^3}}{[2Z^2 - Z + 1]} \\
x[z] &= \frac{2Z^2 + 5Z}{[2Z^2 - Z + 1]} + \frac{\frac{(z^2 + z)}{(z - 1)^3}}{[2Z^2 - Z + 1]} \\
x[z] &= \frac{2Z^2 + 5Z}{[2Z^2 - Z + 1]} + \frac{[z^2 + z]}{[z - 1]^3[2Z^2 - Z + 1]}
\end{aligned}$$

partial fraction:

$$[2Z^2 - Z + 1] = 0$$

$$\begin{aligned}
\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(1)}}{2(2)} \rightarrow Z = \frac{1}{4} \pm \frac{\sqrt{7}}{4}j \\
[2Z^2 - Z + 1] &= \left[Z - \frac{1}{4} \pm \frac{\sqrt{7}}{4}j \right] = \left[Z - \frac{1}{4} - \frac{\sqrt{7}}{4}j \right] \left[Z - \frac{1}{4} + \frac{\sqrt{7}}{4}j \right] \\
&= \left[\left(Z - \frac{1}{4} \right) - \left(\frac{\sqrt{7}}{4}j \right) \right] \left[\left(Z - \frac{1}{4} \right) + \left(\frac{\sqrt{7}}{4}j \right) \right] \\
&= \left\{ \left(Z - \frac{1}{4} \right) \left(Z - \frac{1}{4} \right) + \left(Z - \frac{1}{4} \right) \left(\frac{\sqrt{7}}{4}j \right) - \left(Z - \frac{1}{4} \right) \left(\frac{\sqrt{7}}{4}j \right) - \left(\frac{\sqrt{7}}{4}j \right) \left(\frac{\sqrt{7}}{4}j \right) \right\} \\
&= \left\{ \left(Z - \frac{1}{4} \right) \left(Z - \frac{1}{4} \right) - \left(\frac{\sqrt{7}}{4}j \right) \left(\frac{\sqrt{7}}{4}j \right) \right\} = \left\{ \left(Z - \frac{1}{4} \right)^2 - \left(\frac{\sqrt{7}}{4}j \right)^2 \right\} = \left\{ \left(Z - \frac{1}{4} \right)^2 + \left(\frac{\sqrt{7}}{4} \right)^2 \right\} \\
&= \left\{ \left(Z - \frac{1}{4} \right)^2 + \frac{7}{16} \right\} = \left\{ (4Z - 1)^2 + \frac{7}{16} \right\} = [(4Z - 1)^2 + 0.4375]
\end{aligned}$$

I partial fraction:

$$x[z] = \frac{2Z^2 + 5Z}{[2Z^2 - Z + 1]} = \frac{2Z^2 + 5Z}{\left[Z - \frac{1}{4} - \frac{\sqrt{7}}{4}j \right] \left[Z - \frac{1}{4} + \frac{\sqrt{7}}{4}j \right]} = \frac{Z[2Z + 5]}{[2Z^2 - Z + 1]}$$

$$\begin{aligned}
\frac{x[z]}{z} &= \frac{[2Z + 5]}{[2Z^2 - Z + 1]} = \frac{2Z + 5}{\left[Z - \frac{1}{4} - \frac{\sqrt{7}}{4}j \right] \left[Z - \frac{1}{4} + \frac{\sqrt{7}}{4}j \right]} \\
&= \frac{A}{\left[Z - \frac{1}{4} - \frac{\sqrt{7}}{4}j \right]} + \frac{B}{\left[Z - \frac{1}{4} + \frac{\sqrt{7}}{4}j \right]}
\end{aligned}$$

$$2Z + 5 = A \left[Z - \frac{1}{4} + \frac{\sqrt{7}}{4}j \right] + B \left[Z - \frac{1}{4} - \frac{\sqrt{7}}{4}j \right]$$

when $z = \frac{1}{4} + \frac{\sqrt{7}}{4}j$ we get

$$2 \left[\frac{1}{4} + \frac{\sqrt{7}}{4}j \right] + 5 = A \left[\frac{1}{4} + \frac{\sqrt{7}}{4}j - \frac{1}{4} + \frac{\sqrt{7}}{4}j \right] + B \left[\frac{1}{4} + \frac{\sqrt{7}}{4}j - \frac{1}{4} - \frac{\sqrt{7}}{4}j \right]$$

$$2 \left[\frac{1}{4} + \frac{\sqrt{7}}{4}j \right] + 5 = A \left[\frac{\sqrt{7}}{4}j + \frac{\sqrt{7}}{4}j \right] + B[0]$$

$$\left[\frac{1}{2} + \frac{\sqrt{7}}{2}j \right] + 5 = A \left[\frac{2\sqrt{7}}{4}j \right]$$

$$A \left[\frac{\sqrt{7}}{2}j \right] = \left[\frac{1}{2} + \frac{\sqrt{7}}{2}j \right] + 5$$

$$A = \frac{\left[\frac{1}{2} + \frac{\sqrt{7}}{2}j \right] + 5}{\left[\frac{\sqrt{7}}{2}j \right]} = \frac{\frac{1}{2} + \frac{\sqrt{7}}{2}j + 5}{\left[\frac{\sqrt{7}}{2}j \right]} = \frac{\frac{11}{2} + \frac{\sqrt{7}}{2}j}{\left[\frac{\sqrt{7}}{2}j \right]} = \frac{\frac{11}{2}}{\frac{\sqrt{7}}{2}j} + \frac{\frac{\sqrt{7}}{2}j}{\frac{\sqrt{7}}{2}j} = \frac{11}{2} \times \frac{2}{j\sqrt{7}} + 1$$

$$= \frac{11}{j\sqrt{7}} + 1 = 1 - \frac{11}{7}j\sqrt{7} = 1 - \frac{11\sqrt{7}}{7}j = \frac{8\sqrt{14}}{7}e^{-1,3348j} = 4.28e^{-1,35j}$$

$$|R| = \sqrt{a^2 + b^2} = \sqrt{(1)^2 + \left(\frac{11\sqrt{7}}{7}\right)^2} = \frac{8\sqrt{14}}{7} = 4.27618$$

$$\theta = \tan^{-1} \left[-\frac{\frac{11\sqrt{7}}{7}}{1} \right] = -1.334757034$$

$$A = 1 - \frac{11\sqrt{7}}{7}j = 4.28e^{-1.35j}$$

$$2Z + 5 = A \left[Z - \frac{1}{4} + \frac{\sqrt{7}}{4}j \right] + B \left[Z - \frac{1}{4} - \frac{\sqrt{7}}{4}j \right]$$

$$\text{when } z = \frac{1}{4} - \frac{\sqrt{7}}{4}j \quad \text{we get}$$

$$2 \left[\frac{1}{4} - \frac{\sqrt{7}}{4}j \right] + 5 = A \left[\frac{1}{4} - \frac{\sqrt{7}}{4}j - \frac{1}{4} + \frac{\sqrt{7}}{4}j \right] + B \left[\frac{1}{4} - \frac{\sqrt{7}}{4}j - \frac{1}{4} - \frac{\sqrt{7}}{4}j \right]$$

$$2 \left[\frac{1}{4} - \frac{\sqrt{7}}{4}j \right] + 5 = A[0] + B \left[-\frac{\sqrt{7}}{4}j - \frac{\sqrt{7}}{4}j \right]$$

$$\left[\frac{1}{2} - \frac{\sqrt{7}}{2}j \right] + 5 = B \left[-\frac{2\sqrt{7}}{4}j \right]$$

$$B \left[-\frac{\sqrt{7}}{2}j \right] = \left[\frac{1}{2} - \frac{\sqrt{7}}{2}j \right] + 5$$

$$B = \frac{\left[\frac{1}{2} - \frac{\sqrt{7}}{2}j \right] + 5}{\left[-\frac{\sqrt{7}}{2}j \right]} = -\frac{\frac{1}{2} - \frac{\sqrt{7}}{2}j + 5}{\left[\frac{\sqrt{7}}{2}j \right]} = -\frac{\frac{11}{2} - \frac{\sqrt{7}}{2}j}{\left[\frac{\sqrt{7}}{2}j \right]} = -\frac{\frac{11}{2}}{\frac{\sqrt{7}}{2}j} + \frac{\frac{\sqrt{7}}{2}j}{\frac{\sqrt{7}}{2}j} = -\frac{11}{2} \times \frac{2}{j\sqrt{7}} + 1$$

$$= -\frac{11}{j\sqrt{7}} + 1 = 1 + \frac{11}{7}j\sqrt{7} = 1 + \frac{11\sqrt{7}}{7}j$$

$$B = 1 + \frac{11\sqrt{7}}{7}j = \frac{8\sqrt{14}}{7}e^{1.3348j} = 4.28e^{1.35j}$$

$$\frac{x[z]}{z} = \frac{[2Z + 5]}{[2Z^2 - Z + 1]} = \frac{2Z + 5}{\underbrace{\left[Z - \frac{1}{4} - \frac{\sqrt{7}}{4}j \right]}_A \underbrace{\left[Z - \frac{1}{4} + \frac{\sqrt{7}}{4}j \right]}_B}$$

$$= \frac{\left[Z - \frac{1}{4} - \frac{\sqrt{7}}{4}j \right]}{\left[Z - \frac{1}{4} - \frac{\sqrt{7}}{4}j \right]} + \frac{\left[Z - \frac{1}{4} + \frac{\sqrt{7}}{4}j \right]}{\left[Z - \frac{1}{4} + \frac{\sqrt{7}}{4}j \right]}$$

$$x[z] = \frac{Z[2Z + 5]}{[2Z^2 - Z + 1]} = \frac{Z[2Z + 5]}{\underbrace{\left[Z - \frac{1}{4} - \frac{\sqrt{7}}{4}j \right]}_{AZ} \underbrace{\left[Z - \frac{1}{4} + \frac{\sqrt{7}}{4}j \right]}_{BZ}}$$

$$= \frac{\left[Z - \frac{1}{4} - \frac{\sqrt{7}}{4}j \right]}{\left[Z - \frac{1}{4} - \frac{\sqrt{7}}{4}j \right]} + \frac{\left[Z - \frac{1}{4} + \frac{\sqrt{7}}{4}j \right]}{\left[Z - \frac{1}{4} + \frac{\sqrt{7}}{4}j \right]}$$

$$\begin{aligned}
x[z] &= \frac{AZ}{\left[Z - \frac{1}{4} - \frac{\sqrt{7}}{4}j\right]} + \frac{BZ}{\left[Z - \frac{1}{4} + \frac{\sqrt{7}}{4}j\right]} = \frac{\left[1 - \frac{11\sqrt{7}}{7}j\right]Z}{\left[Z - \frac{1}{4} - \frac{\sqrt{7}}{4}j\right]} + \frac{\left[1 + \frac{11\sqrt{7}}{7}j\right]Z}{\left[Z - \frac{1}{4} + \frac{\sqrt{7}}{4}j\right]} \\
x[z] &= \frac{\left[1 - \frac{11\sqrt{7}}{7}j\right]Z}{\left[Z - \frac{1}{4} - \frac{\sqrt{7}}{4}j\right]} + \frac{\left[1 + \frac{11\sqrt{7}}{7}j\right]Z}{\left[Z - \frac{1}{4} + \frac{\sqrt{7}}{4}j\right]} \\
&= \left[1 - \frac{11\sqrt{7}}{7}j\right] \frac{Z}{\left[Z - \frac{1}{4} - \frac{\sqrt{7}}{4}j\right]} + \left[1 + \frac{11\sqrt{7}}{7}j\right] \frac{Z}{\left[Z - \frac{1}{4} + \frac{\sqrt{7}}{4}j\right]} \\
x[z] &= \left[\frac{8\sqrt{14}}{7}e^{-1,3348j}\right] \frac{Z}{\left[Z - \frac{1}{4} - \frac{\sqrt{7}}{4}j\right]} + \left[\frac{8\sqrt{14}}{7}e^{1,3348j}\right] \frac{Z}{\left[Z - \frac{1}{4} + \frac{\sqrt{7}}{4}j\right]}
\end{aligned}$$

the inverse of the Z transform: $z^{-1}x[z] = x[k]$

$$\begin{aligned}
x[k] &= z^{-1}x[z] = z^{-1} \left\{ \left[\frac{8\sqrt{14}}{7}e^{-1,3348j}\right] \frac{Z}{\left[Z - \frac{1}{4} - \frac{\sqrt{7}}{4}j\right]} + \left[\frac{8\sqrt{14}}{7}e^{1,3348j}\right] \frac{Z}{\left[Z - \frac{1}{4} + \frac{\sqrt{7}}{4}j\right]} \right\} \\
x[k] &= z^{-1} \left\{ \left[\frac{8\sqrt{14}}{7}e^{-1,3348j}\right] \frac{Z}{\left[Z - \frac{1}{4} - \frac{\sqrt{7}}{4}j\right]} \right\} + z^{-1} \left\{ \left[\frac{8\sqrt{14}}{7}e^{1,3348j}\right] \frac{Z}{\left[Z - \frac{1}{4} + \frac{\sqrt{7}}{4}j\right]} \right\} \\
x[k] &= \left[\frac{8\sqrt{14}}{7}e^{-1,3348j}\right] z^{-1} \left\{ \frac{Z}{\left[Z - \frac{1}{4} - \frac{\sqrt{7}}{4}j\right]} \right\} + \left[\frac{8\sqrt{14}}{7}e^{1,3348j}\right] z^{-1} \left\{ \frac{Z}{\left[Z - \frac{1}{4} + \frac{\sqrt{7}}{4}j\right]} \right\} \\
x[k] &= \left[\frac{8\sqrt{14}}{7}e^{-1,3348j}\right] \left[e^{-\left(-\frac{1}{4} - \frac{\sqrt{7}}{4}j\right)} \right] + \left[\frac{8\sqrt{14}}{7}e^{1,3348j}\right] \left[e^{-\left(-\frac{1}{4} + \frac{\sqrt{7}}{4}j\right)} \right] \\
x[k] &= \left[\frac{8\sqrt{14}}{7}e^{-1,3348j}\right] \left[e^{\frac{1}{4} + \frac{\sqrt{7}}{4}j} \right] + \left[\frac{8\sqrt{14}}{7}e^{1,3348j}\right] \left[e^{\frac{1}{4} - \frac{\sqrt{7}}{4}j} \right] \\
x[k] &= \left[\frac{8\sqrt{14}}{7}e^{-1,3348j}\right] \left[e^{\frac{1}{4}} e^{\frac{\sqrt{7}}{4}j} \right] + \left[\frac{8\sqrt{14}}{7}e^{1,3348j}\right] \left[e^{\frac{1}{4}} e^{-\frac{\sqrt{7}}{4}j} \right] \\
x[k] &= e^{\frac{1}{4}} \left\{ \left[\frac{8\sqrt{14}}{7}e^{-1,3348j}\right] \left[e^{\frac{\sqrt{7}}{4}j} \right] + \left[\frac{8\sqrt{14}}{7}e^{1,3348j}\right] \left[e^{-\frac{\sqrt{7}}{4}j} \right] \right\} \\
x[k] &= e^{\frac{1}{4}} \left\{ \left[\frac{8\sqrt{14}}{7}\right] \left[e^{-1,3348j} e^{\frac{\sqrt{7}}{4}j} \right] + \left[\frac{8\sqrt{14}}{7}\right] \left[e^{1,3348j} e^{-\frac{\sqrt{7}}{4}j} \right] \right\} \\
x[k] &= e^{\frac{1}{4}} \left[\frac{8\sqrt{14}}{7}\right] \left\{ \left[e^{-1,3348j} e^{\frac{\sqrt{7}}{4}j} \right] + \left[e^{1,3348j} e^{-\frac{\sqrt{7}}{4}j} \right] \right\}
\end{aligned}$$

$$\begin{aligned}
x[k] &= e^{\frac{1}{4}} \left[\frac{8\sqrt{14}}{7} \right] \left\{ \left[e^{-1,3348j + \frac{\sqrt{7}}{4}j} \right] + \left[e^{1,3348j - \frac{\sqrt{7}}{4}j} \right] \right\} \\
x[k] &= e^{\frac{1}{4}} \left[\frac{8\sqrt{14}}{7} \right] \left\{ \left[e^{\left(-1,3348j + \frac{\sqrt{7}}{4}j\right)} \right] + \left[e^{\left(1,3348j - \frac{\sqrt{7}}{4}j\right)} \right] \right\} \\
x[k] &= e^{\frac{1}{4}} \left[\frac{8\sqrt{14}}{7} \right] \left\{ \left[e^{\left(-1,3348 + \frac{\sqrt{7}}{4}\right)j} \right] + \left[e^{\left(1,3348 - \frac{\sqrt{7}}{4}\right)j} \right] \right\} \\
x[k] &= e^{\frac{1}{4}} \left[\frac{8\sqrt{14}}{7} \right] \left\{ e^{\left(-1,3348 + \frac{\sqrt{7}}{4}\right)j} + e^{\left(1,3348 - \frac{\sqrt{7}}{4}\right)j} \right\} \\
x[k] &= e^{\frac{1}{4}} \left[\frac{8\sqrt{14}}{7} \right] \left\{ e^{\left(-1,3348 + \frac{\sqrt{7}}{4}\right)j} + e^{-\left(-1,3348 + \frac{\sqrt{7}}{4}\right)j} \right\} \\
x[k] &= e^{\frac{1}{4}} \left[\frac{8\sqrt{14}}{7} \right] \left\{ 2 \cos \left(-1,3348 + \frac{\sqrt{7}}{4} \right) \right\} \\
x[k] &= e^{\frac{1}{4}} \left[\frac{16\sqrt{14}}{7} \right] \left\{ \cos \left(-1,3348 + \frac{\sqrt{7}}{4} \right) \right\} \\
x[k] &= e^{\frac{1}{4}} \left[\frac{16\sqrt{14}}{7} \right] \{ \cos(-0.6733192062) \} \\
x[k] &= 10.98144728 \{ \cos(-0.6733192062) \} \\
x[k] &= 10.98144728 \cos(-0.6733192062) \\
x[k] &= 10.98144728 \cos(0.6733192062) \\
x[k] &= 10.98068901 \cong 11 \\
x[k] &\cong 11
\end{aligned}$$

II partial fraction

$$\begin{aligned}
x[z] &= \frac{[z^2 + z]}{[z - 1]^3 [2 Z^2 - Z + 1]} = \frac{z[z + 1]}{[z - 1]^3 [2 Z^2 - Z + 1]} \\
&= \frac{z[z + 1]}{[z - 1]^3 \left[Z - \frac{1}{4} - \frac{\sqrt{7}}{4}j \right] \left[Z - \frac{1}{4} + \frac{\sqrt{7}}{4}j \right]} \\
\frac{x[z]}{z} &= \frac{[z + 1]}{[z - 1]^3 [2 Z^2 - Z + 1]} = \frac{AZ + B}{[2 Z^2 - Z + 1]} + \frac{C}{[z - 1]} + \frac{D}{[z - 1]^2} + \frac{E}{[z - 1]^3} \\
\frac{x[z]}{z} &= \frac{A}{\left[Z - \frac{1}{4} - \frac{\sqrt{7}}{4}j \right]} + \frac{B}{\left[Z - \frac{1}{4} + \frac{\sqrt{7}}{4}j \right]} + \frac{C}{[z - 1]} + \frac{D}{[z - 1]^2} + \frac{E}{[z - 1]^3} \\
&= \frac{[z + 1]}{[z - 1]^3 [2 Z^2 - Z + 1]} \\
x[z] &= \frac{Az}{\left[Z - \frac{1}{4} - \frac{\sqrt{7}}{4}j \right]} + \frac{Bz}{\left[Z - \frac{1}{4} + \frac{\sqrt{7}}{4}j \right]} + \frac{Cz}{[z - 1]} + \frac{Dz}{[z - 1]^2} + \frac{Ez}{[z - 1]^3}
\end{aligned}$$

the inverse of the Z transform: $z^{-1}x[z] = x[k]$

$$x[k] = z^{-1}x[z] = \left\{ \frac{Az}{\left[Z - \frac{1}{4} - \frac{\sqrt{7}}{4}j\right]} + \frac{Bz}{\left[Z - \frac{1}{4} + \frac{\sqrt{7}}{4}j\right]} + \frac{Cz}{[z-1]} + \frac{Dz}{[z-1]^2} + \frac{Ez}{[z-1]^3} \right\}$$

$$x[k] = z^{-1}x[z]$$

$$= z^{-1} \left\{ \frac{Az}{\left[Z - \frac{1}{4} - \frac{\sqrt{7}}{4}j\right]} \right\} + z^{-1} \left\{ \frac{Bz}{\left[Z - \frac{1}{4} + \frac{\sqrt{7}}{4}j\right]} \right\} + z^{-1} \left\{ \frac{Cz}{[z-1]} \right\} \\ + z^{-1} \left\{ \frac{Dz}{[z-1]^2} \right\} + z^{-1} \left\{ \frac{Ez}{[z-1]^3} \right\}$$

$$x[k] = z^{-1}x[z]$$

$$= Az^{-1} \left\{ \frac{z}{\left[Z - \frac{1}{4} - \frac{\sqrt{7}}{4}j\right]} \right\} + Bz^{-1} \left\{ \frac{z}{\left[Z - \frac{1}{4} + \frac{\sqrt{7}}{4}j\right]} \right\} + Cz^{-1} \left\{ \frac{z}{[z-1]} \right\} \\ + Dz^{-1} \left\{ \frac{z}{[z-1]^2} \right\} + Ez^{-1} \left\{ \frac{z}{[z-1]^3} \right\}$$

$$x[k] = z^{-1}x[z] = Ae^{-\left(\frac{1}{4} - \frac{\sqrt{7}}{4}j\right)} + Be^{-\left(\frac{1}{4} + \frac{\sqrt{7}}{4}j\right)} + Ce^{-(-1)} + Dk(-1)^{k-1} + E\{k^2 - k\}$$

$$x[k] = Ae^{\left(\frac{1}{4} + \frac{\sqrt{7}}{4}j\right)} + Be^{\left(\frac{1}{4} - \frac{\sqrt{7}}{4}j\right)} + Ce + Dk(-1)^{k-1} + E\{k^2 - k\}$$

$$x[k] = \left[Ae^{\left(\frac{1}{4} + \frac{\sqrt{7}}{4}j\right)} + Be^{\left(\frac{1}{4} - \frac{\sqrt{7}}{4}j\right)} \right] + Ce + Dk(-1)^{k-1} + E\{k^2 - k\}$$

$$x[k] = \left[Ae^{\frac{1}{4} + \frac{\sqrt{7}}{4}j} + Be^{\frac{1}{4} - \frac{\sqrt{7}}{4}j} \right] + Ce + Dk(-1)^{k-1} + E\{k^2 - k\}$$

$$x[k] = \left[Ae^{\frac{1}{4}} e^{\frac{\sqrt{7}}{4}j} + Be^{\frac{1}{4}} e^{-\frac{\sqrt{7}}{4}j} \right] + Ce + Dk(-1)^{k-1} + E\{k^2 - k\}$$

$$x[k] = e^{\frac{1}{4}} \left[Ae^{\frac{\sqrt{7}}{4}j} + Be^{-\frac{\sqrt{7}}{4}j} \right] + Ce + Dk(-1)^{k-1} + E\{k^2 - k\}$$

$$x[k] = e^{\frac{1}{4}} [A \cong B] \left[e^{\frac{\sqrt{7}}{4}j} + e^{-\frac{\sqrt{7}}{4}j} \right] + Ce + Dk(-1)^{k-1} + E\{k^2 - k\}$$

$$[A \cong B] = N \quad \text{for some reasons that you don't need to know}$$

$$x[k] = e^{\frac{1}{4}} [N] \left[e^{\frac{\sqrt{7}}{4}j} + e^{-\frac{\sqrt{7}}{4}j} \right] + Ce + Dk(-1)^{k-1} + E\{k^2 - k\}$$

$$x[k] = e^{\frac{1}{4}} [N] \left[2 \cos \frac{\sqrt{7}}{4} \right] + Ce + Dk(-1)^k (-1)^{-1} + E\{k^2 - k\}$$

$$x[k] = 2e^{\frac{1}{4}} [N] \left[\cos \frac{\sqrt{7}}{4} \right] + Ce - Dk(-1)^k + E\{k^2 - k\}$$

$$x[k] = 2e^{\frac{1}{4}} [N] + Ce - Dk(-1)^k + E\{k^2 - k\}$$

$$x[k] = 2.568[N] + Ce - Dk(-1)^k + E\{k^2 - k\}$$

$$x[k] = 2.568[N] + Ce - Dk(-1)^k + E\{k^2 - k\}$$

inverse partial I + inverse partial II

$$\begin{aligned} x[k] &= 2.568[N] + Ce - Dk(-1)^k + E\{k^2 - k\} + 11 \\ x[k] &= [11 + 2.568][N] + Ce - Dk(-1)^k + E\{k^2 - k\} \\ x[k] &= 13.568[N] + Ce - Dk(-1)^k + E\{k^2 - k\} \\ x[k] &= 14[N] + Ce - Dk(-1)^k + E\{k^2 - k\} \end{aligned}$$

$$x[k] = 14[N] + Ce - Dk(-1)^k + E\{k^2 - k\}$$

3. Write down the first five terms of the sequence defined by $f[k] = 4(2)^{k-1}u[k - 1]$, $k \geq 0$ Find its Z transform directly, and also by using the second shift theorem.

$$f[k] = 4(2)^{k-1}u[k - 1], \quad k \geq 0$$

$$f[k] = 4(2)^{k-1}u[k - 1] = \begin{cases} 4(2)^{k-1} & \dots \dots \dots k \geq 0 \\ 0 & \dots \dots \dots k < 0 \end{cases}$$

$$Z\{a^k u[k]\} = Z\{4(2)^{k-1}u[k - 1]\} = \sum_{k=0}^{\infty} 4(2)^{k-1}Z^{1-k} = \sum_{k=0}^{\infty} \{4(2)Z^{-1}\}^{k-1}$$

$$k = 0, 1, 2, 3 \dots$$

$$Z\{4(2)^{k-1}u[k - 1]\} = \{4(2)Z^{-1}\}^{0-1} + \{4(2)Z^{-1}\}^{1-1} + \{4(2)Z^{-1}\}^{2-1} + \{4(2)Z^{-1}\}^{3-1} + \dots + \{4(2)Z^{-1}\}^k$$

the sequence

$$Z\{4(2)^{k-1}u[k - 1]\} = \{4(2)Z^{-1}\}^{-1} + \{4(2)Z^{-1}\}^0 + \{4(2)Z^{-1}\}^1 + \{4(2)Z^{-1}\}^2 + \dots + \{4(2)Z^{-1}\}^k$$

Geometric – Series: $r = az^{-1}$

$$S[k] = \frac{1}{1 - 4[2]z^{-1}} = \frac{1}{1 - \frac{4[2]}{z}} = \frac{1}{\frac{z - 4[2]}{z}} = \frac{z}{z - 4[2]}$$

4. Solve the difference equation $y[k + 2] + 3y[k + 1] + 2y[k] = 0$

subject to the condition $y[0] = 0, \quad y[1] = 1$

$$y[k + 2] + 3y[k + 1] + 2y[k] = 0$$

$$Z\{y[k + 2] + 3y[k + 1] + 2y[k]\} = Z\{0\}$$

Z transform Table:

$$y[k] = y[z]$$

$$y[k + 1] = \{Z y[z] - y[0]\} = \{Z y[z] - 0\} = \{Z y[z]\}$$

$$y[k + 2] = \{Z^2 y[z] - Z^2 y[0] - Zy[1]\} = \{Z^2 y[z] - Z^2 [0] - Z[1]\} = \{Z^2 y[z] - Z\}$$

$$Z\{y[k + 2] + 3y[k + 1] + 2y[k]\} = Z\{0\}$$

$$Z\{y[k + 2]\} + Z\{3y[k + 1]\} + Z\{2y[k]\} = Z\{0\}$$

$$Z\{y[k + 2]\} + 3Z\{y[k + 1]\} + 2Z\{y[k]\} = Z\{0\}$$

$$\{Z^2 y[z] - Z\} + 3\{Z y[z]\} + 2\{y[z]\} = \{0\}$$

$$Z^2 y[z] - Z + 3 Z y[z] + 2y[z] = 0$$

$$Z^2 y[z] + 3 Z y[z] + 2y[z] = Z$$

$$\{Z^2 y[z] + 3 Z y[z] + 2y[z]\} = Z$$

$$y[z][Z^2 + 3 Z + 2] = Z$$

$$y[z] = \frac{Z}{[Z^2 + 3 Z + 2]}$$

partial fraction:

$$y[z] = \frac{Z}{[Z^2 + 3 Z + 2]} = \frac{Z}{[Z + 1][Z + 2]}$$

$$[Z^2 + 3 Z + 2] = 0$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{(3)^2 - 4(1)(2)}}{2(1)} \rightarrow \quad Z_1 = -1, \quad Z_2 = -2$$

$$[Z^2 + 3 Z + 2] = [Z + 1][Z + 2]$$

$$y[z] = \frac{Z}{[Z^2 + 3 Z + 2]} = \frac{Z}{[Z + 1][Z + 2]} = \frac{AZ}{[Z + 1]} + \frac{BZ}{[Z + 2]}$$

$$\frac{y[z]}{Z} = \frac{1}{[Z^2 + 3 Z + 2]} = \frac{1}{[Z + 1][Z + 2]} = \frac{A}{[Z + 1]} + \frac{B}{[Z + 2]}$$

$$1 = A[Z + 2] + B[Z + 1]$$

when $z = -1$ we get $1 = A[(-1) + 2] + B[(-1) + 1]$

$$1 = A[-1 + 2] + B[-1 + 1]$$

$$1 = A[1] + B[0]$$

$$A = 1$$

$$1 = A[Z + 2] + B[Z + 1]$$

when $z = -2$ we get $1 = A[(-2) + 2] + B[(-2) + 1]$

$$1 = A[-2 + 2] + B[-2 + 1]$$

$$1 = A[0] + B[-1]$$

$$1 = B[-1]$$

$$B = -1$$

$$y[z] = \frac{Z}{[Z + 1][Z + 2]} = \frac{AZ}{[Z + 1]} + \frac{BZ}{[Z + 2]} = \frac{[1]Z}{[Z + 1]} + \frac{[-1]Z}{[Z + 2]} = \frac{Z}{[Z + 1]} - \frac{Z}{[Z + 2]}$$

the inverse of the Z transform: $z^{-1}y[z] = y[k]$

$$y[k] = z^{-1}y[z] = z^{-1} \left\{ \frac{Z}{[Z + 1]} - \frac{Z}{[Z + 2]} \right\} = z^{-1} \left\{ \frac{Z}{[Z + 1]} \right\} - z^{-1} \left\{ \frac{Z}{[Z + 2]} \right\}$$

$$y[k] = (-1)^k - (-2)^k$$

5. Find the inverse Z transform of $\frac{z^2 + 3z}{3z^2 + 2z - 5}$

partial fraction:

$$G(z) = \frac{z^2 + 3z}{3z^2 + 2z - 5} = \frac{z[z + 3]}{3z^2 + 2z - 5}$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{(2)^2 - 4(3)(-5)}}{2(3)} \rightarrow Z_1 = 1, \quad Z_2 = -\frac{5}{3}$$

$$3z^2 + 2z - 5 = [z - 1] \left[z + \frac{5}{3} \right]$$

$$G(z) = \frac{z^2 + 3z}{3z^2 + 2z - 5} = \frac{z^2 + 3z}{[z - 1] \left[z + \frac{5}{3} \right]} = \frac{z[z + 3]}{3z^2 + 2z - 5}$$

$$\frac{G(z)}{z} = \frac{[z + 3]}{3z^2 + 2z - 5} = \frac{[z + 3]}{[z - 1] \left[z + \frac{5}{3} \right]} = \frac{A}{[z - 1]} + \frac{B}{\left[z + \frac{5}{3} \right]}$$

$$[z + 3] = A \left[z + \frac{5}{3} \right] + B[z - 1]$$

when $z = 1$ we get $[(1) + 3] = A \left[(1) + \frac{5}{3} \right] + B[(1) - 1]$

$$[1 + 3] = A \left[1 + \frac{5}{3} \right] + B[1 - 1]$$

$$[4] = A \left[\frac{8}{3} \right] + B[0]$$

$$[4] = A \left[\frac{8}{3} \right]$$

$$A = \frac{4}{\frac{8}{3}} = 4 \times \frac{3}{8} = \frac{3}{2} = 1.5$$

$$[z + 3] = A \left[z + \frac{5}{3} \right] + B[z - 1]$$

when $z = -\frac{5}{3}$ we get $\left[\left(-\frac{5}{3} \right) + 3 \right] = A \left[\left(-\frac{5}{3} \right) + \frac{5}{3} \right] + B \left[\left(-\frac{5}{3} \right) - 1 \right]$

$$\left[-\frac{5}{3} + 3 \right] = A \left[-\frac{5}{3} + \frac{5}{3} \right] + B \left[-\frac{5}{3} - 1 \right]$$

$$\left[\frac{4}{3} \right] = A[0] + B \left[-\frac{8}{3} \right]$$

$$\left[\frac{4}{3} \right] = B \left[-\frac{8}{3} \right]$$

$$B = -\frac{\frac{4}{3}}{\frac{8}{3}} = -\frac{4}{3} \times \frac{3}{8} = -\frac{4}{8} = -\frac{1}{2} = -0.5$$

$$\frac{G(z)}{z} = \frac{[z + 3]}{3z^2 + 2z - 5} = \frac{[z + 3]}{[z - 1] \left[z + \frac{5}{3} \right]} = \frac{A}{[z - 1]} + \frac{B}{\left[z + \frac{5}{3} \right]}$$

$$\frac{G(z)}{z} = \frac{A}{[z - 1]} + \frac{B}{\left[z + \frac{5}{3} \right]}$$

$$G(z) = \frac{Az}{[z - 1]} + \frac{Bz}{\left[z + \frac{5}{3} \right]} = \frac{\frac{3}{2}z}{[z - 1]} + \frac{-\frac{1}{2}z}{\left[z + \frac{5}{3} \right]} = \frac{3}{2} \frac{z}{[z - 1]} - \frac{1}{2} \frac{z}{\left[z + \frac{5}{3} \right]}$$

the inverse of the Z transform: $z^{-1}G[z] = G[k]$

$$G(z) = \frac{3}{2} \frac{z}{[z-1]} - \frac{1}{2} \frac{z}{\left[z + \frac{5}{3}\right]}$$

$$G[k] = z^{-1}G[z] = z^{-1} \left\{ \frac{3}{2} \frac{z}{[z-1]} - \frac{1}{2} \frac{z}{\left[z + \frac{5}{3}\right]} \right\} = \frac{3}{2} z^{-1} \left\{ \frac{z}{[z-1]} \right\} - \frac{1}{2} z^{-1} \left\{ \frac{z}{\left[z + \frac{5}{3}\right]} \right\}$$

$$G[k] = \frac{3}{2} (h_k) - \frac{1}{2} \left(-\frac{5}{3}\right)^k$$

$$G[k] = 1.5 (h_k) - 0.5 \left(-\frac{5}{3}\right)^k$$

FISA

1. Find the directions in which the directional derivative of $f(x, y) = e^x \sin y$, at the point $(0, \frac{\pi}{3})$ has the value 1.

$$D_{\vec{u}} f = 1$$

$$f_x = e^x \sin y$$

$$f_y = e^x \cos y$$

$$\nabla f_{(x,y)} = (e^x \sin y)i + (e^x \cos y)j = e^x (\sin y)i + (\cos y)j$$

$$\text{at the point } (0, \frac{\pi}{3})$$

$$x = 0$$

$$y = \frac{\pi}{3}$$

$$f_x = e^x \sin y = e^0 \sin \frac{\pi}{3} = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$f_y = e^x \cos y = e^0 \cos \frac{\pi}{3} = \cos \frac{\pi}{3} = \frac{1}{2}$$

$$\nabla f_{(x,y)} = \left(\frac{\sqrt{3}}{2}\right)i + \left(\frac{1}{2}\right)j = \frac{1}{2}[\sqrt{3}i + j]$$

$$D_{\vec{u}} f = \nabla f_{(x,y)} \cdot \vec{u}$$

$$1 = \frac{1}{2}[\sqrt{3}i + j] \cdot \vec{u}$$

$$2 = [\sqrt{3}i + j] \cdot \vec{u}$$

$$\frac{2}{[\sqrt{3}i + j]} = \vec{u}$$

$$\vec{u} = \frac{2}{[\sqrt{3}i + j]} = \frac{2[\sqrt{3}i - j]}{[\sqrt{3}i + j][\sqrt{3}i - j]} = \frac{2[\sqrt{3}i - j]}{\{[\sqrt{3}i]^2 - [j]^2\}} = \frac{2[\sqrt{3}i - j]}{\{-3 - [-1]\}} = \frac{2[\sqrt{3}i - j]}{[-3 + 1]} = \frac{2[\sqrt{3}i - j]}{[-2]}$$

$$\vec{u} = \frac{2[\sqrt{3}i - j]}{[-2]} = -[\sqrt{3}i - j] = [-\sqrt{3}i + j]$$

$$\vec{u} = [-\sqrt{3}i + j] = [-ai + bj] = 2 < -\frac{\pi}{6} = 2e^{-\frac{\pi}{6}i} = 2e^{-30^\circ i}$$

$$|R| = \sqrt{a^2 + b^2} = \sqrt{(\sqrt{3})^2 + (1)^2} = \sqrt{4} = 2$$

$$\theta = \tan^{-1} \frac{b}{a} = \tan^{-1} \frac{1}{-\sqrt{3}} = -\frac{\pi}{6} = -30^\circ$$

our the directions:

the angles is $\theta = -30^\circ$ and the magnitude is 2

2. Use Green's theorem to evaluate

$$\oint_c \vec{F} \cdot d\vec{r}$$

where $F(x, y) = (y \cos x - x y \sin x) \vec{i} + (xy + x \cos x) \vec{j}$,
 C is the triangle form $(0,0)$ to $(0,4)$ to $(2,0)$ to $(0,0)$.

$$F(x, y) = (y \cos x - x y \sin x) \vec{i} + (xy + x \cos x) \vec{j}$$

$$\nabla = \begin{vmatrix} (y \cos x - x y \sin x) & (xy + x \cos x) \\ D_X & D_Y \end{vmatrix}$$

$$\nabla = -D_Y(y \cos x - x y \sin x) + D_X(xy + x \cos x)$$

$$\nabla = -(\cos x - x \sin x) + [y + D_X(x \cos x)]$$

$$\nabla = -\cos x + x \sin x + y + (\cos x - x \sin x)$$

$$\nabla = -\cos x + x \sin x + y + \cos x - x \sin x$$

$$\nabla = y$$

the triangle form $(0,0)$ to $(0,4)$ to $(2,0)$ to $(0,0)$

$$A = D = (0,0)$$

$$B = (0,4)$$

$$C = (2,0)$$

$$(2,0) \text{ to } (0,4)$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 0}{0 - 2} = -2$$

$$(2,0)$$

$$y - y_1 = m(x - x_1)$$

$$y - 0 = -2(x - 2)$$

$$y = -2(x - 2) = -2x + 4$$

$$(0,4)$$

$$y - y_2 = m(x - x_2)$$

$$y - 4 = -2(x - 0)$$

$$y = -2x + 4$$

$$\begin{aligned} \oint &= \int_{x_1}^{x_2} \int_{y_1}^{y_2} \nabla dx dy = \int_0^2 \int_0^{(-2x+4)} y dx dy = \int_0^2 \left[\frac{y^2}{2} \right] dx = \int_0^2 \left[\frac{(-2x+4)^2 - (0)^2}{2} \right] dx \\ &= \int_0^2 \left[\frac{(-2x+4)^2}{2} \right] dx = \int_0^2 \left[\frac{(4x^2 - 16x + 16)}{2} \right] dx \\ &= \frac{1}{2} \int_0^2 [4x^2 - 16x + 16] dx = \frac{1}{2} \left[\frac{4}{3} x^3 - \frac{16}{2} x^2 + 16x \right] = \frac{16}{3} \end{aligned}$$

$$(-2x + 4)^2 = (-2x + 4)(-2x + 4) = 4x^2 - 8x - 8x + 16 = 4x^2 - 16x + 16$$

$$\oint = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \nabla dx dy = \int_0^2 \int_0^{(-2x+4)} y dx dy = \frac{16}{3}$$

3. In the circuit in figure 1, a battery supplies a voltage of $E(t) = 40 \sin 60t$ volts, the inductance is $1H$, the resistance is 20 ohms and $I(0) = 1A$.

(a) Find $I(t)$.

(b) Find the current after 0.1 seconds.

$$E = V_r + V_l$$

$$E = I_1 R_1 + L \frac{dI_1}{dx}$$

$$40 \sin 60t = 20I_1 + \frac{dI_1}{dx}$$

$$\frac{dI_1}{dx} + 20I_1 = 40 \sin 60t$$

$$\mu = e^{\int 20dt} = e^{20t}$$

$$I(t) = I_1$$

$$\begin{aligned} e^{20t} I(t) &= \int (e^{20t} 40 \sin 60t) dt = 40 \int (e^{20t} \sin 60t) dt = 40 \int e^{20t} (\sin 60t) dt \\ &= 40 \int e^{20t} \left[\frac{e^{j60t} - e^{-j60t}}{2j} \right] dt \\ &= 40 \int \left[\frac{e^{j60t} e^{20t} - e^{-j60t} e^{20t}}{2j} \right] dt = 40 \int \left[\frac{e^{j60t+20t} - e^{-j60t+20t}}{2j} \right] dt \\ &= 40 \int \left[\frac{e^{(j60+20)t} - e^{(-j60+20)t}}{2j} \right] dt = 40 \int \left[\frac{e^{(j60+20)t} - e^{(20-j60)t}}{2j} \right] dt \\ &= 40 \int \left[\frac{e^{(j60+20)t} - e^{(20-j60)t}}{2j} \right] dt = \frac{40}{2j} \int [e^{(j60+20)t} - e^{(20-j60)t}] dt \\ &= \frac{20}{j} \int [e^{(j60+20)t} - e^{(20-j60)t}] dt = \frac{20}{j} \left[\frac{e^{(j60+20)t}}{(j60+20)} - \frac{e^{(20-j60)t}}{(20-j60)} \right] + C \\ &= \frac{20}{j} \left[\frac{e^{(j60+20)t}}{(j60+20)} \right] - \frac{20}{j} \left[\frac{e^{(20-j60)t}}{(20-j60)} \right] + C = 20 \left[\frac{e^{(j60+20)t}}{(j20-60)} \right] - 20 \left[\frac{e^{(20-j60)t}}{(j20+60)} \right] + C \\ &= \left[\frac{20}{(j20-60)} \right] e^{(j60+20)t} - \left[\frac{20}{(j20+60)} \right] e^{(20-j60)t} + C \\ &= - \left[\frac{20}{(60-j20)} \right] e^{(j60+20)t} - \left[\frac{20}{(j20+60)} \right] e^{(20-j60)t} + C \\ &= - \frac{20}{20} \left[\frac{1}{(3-j)} \right] e^{(j60+20)t} - \frac{20}{20} \left[\frac{1}{(j+3)} \right] e^{(20-j60)t} + C \\ &= - \left[\frac{1}{(3-j)} \right] e^{(j60+20)t} - \left[\frac{1}{(j+3)} \right] e^{(20-j60)t} + C \\ &= - \left[\frac{(3+j)}{10} \right] e^{(j60+20)t} - \left[\frac{(3-j)}{10} \right] e^{(20-j60)t} + C \end{aligned}$$

$$\begin{aligned} \left[\frac{1}{(3-j)} \right] &= \left[\frac{1}{(3-j)} \right] \left[\frac{(3+j)}{(3+j)} \right] = \left[\frac{(3+j)}{(3-j)(3+j)} \right] = \left[\frac{(3+j)}{(3)^2 - (j)^2} \right] = \left[\frac{(3+j)}{9 - (-1)} \right] \\ &= \left[\frac{(3+j)}{9+1} \right] = \left[\frac{(3+j)}{10} \right] = \frac{(3+j)}{10} \\ \left[\frac{1}{(j+3)} \right] &= \left[\frac{1}{(j+3)} \right] \left[\frac{(3-j)}{(3-j)} \right] = \left[\frac{(3-j)}{(3-j)(j+3)} \right] = \left[\frac{(3-j)}{(3)^2 - (j)^2} \right] = \left[\frac{(3-j)}{9 - (-1)} \right] \\ &= \left[\frac{(3-j)}{9+1} \right] = \left[\frac{(3-j)}{10} \right] = \frac{(3-j)}{10} \end{aligned}$$

$$\begin{aligned} e^{20t} I(t) &= - \left[\frac{(3+j)}{10} \right] e^{(j60+20)t} - \left[\frac{(3-j)}{10} \right] e^{(20-j60)t} + C \\ &= - \left[\frac{(3+j)}{10} \right] e^{(j60)t} e^{(20)t} - \left[\frac{(3-j)}{10} \right] e^{(-j60)t} e^{(20)t} + C \\ &= -e^{(20)t} \left\{ \left[\frac{(3+j)}{10} \right] e^{(j60)t} + \left[\frac{(3-j)}{10} \right] e^{(-j60)t} \right\} + C \end{aligned}$$

$$\begin{aligned}
e^{20t} I(t) &= -e^{(20)t} \left[\left[\frac{(3+j)}{10} \right] e^{(j60)t} + \left[\frac{(3-j)}{10} \right] e^{(-j60)t} \right] + C \\
\frac{e^{20t}}{e^{20t}} I(t) &= -\frac{e^{(20)t}}{e^{20t}} \left[\left[\frac{(3+j)}{10} \right] e^{(j60)t} + \left[\frac{(3-j)}{10} \right] e^{(-j60)t} \right] + \frac{C}{e^{20t}} \\
I(t) &= -\left[\left[\frac{(3+j)}{10} \right] e^{(j60)t} + \left[\frac{(3-j)}{10} \right] e^{(-j60)t} \right] + \frac{C}{e^{20t}} \\
I(t) &= -\left[\left[\frac{(3+j)}{10} \right] e^{(j60)t} + \left[\frac{(3-j)}{10} \right] e^{(-j60)t} \right] + C e^{-20t} \\
I(t) &= -\left[\left[\frac{3}{10} + \frac{j}{10} \right] [\cos 60t + j \sin 60t] + \left[\frac{3}{10} - \frac{j}{10} \right] [\cos 60t - j \sin 60t] \right] + C e^{-20t} \\
I(t) &= -\left[\left[\frac{3}{10} \cos 60t + \frac{j}{10} \cos 60t + \frac{3}{10} j \sin 60t - \frac{1}{10} \sin 60t \right] \right. \\
&\quad \left. + \left[\frac{3}{10} \cos 60t - \frac{j}{10} \cos 60t - \frac{3}{10} j \sin 60t - \frac{1}{10} \sin 60t \right] \right] + C e^{-20t} \\
I(t) &= -\left[\frac{3}{10} \cos 60t + \frac{j}{10} \cos 60t + \frac{3}{10} j \sin 60t - \frac{1}{10} \sin 60t + \frac{3}{10} \cos 60t \right. \\
&\quad \left. - \frac{j}{10} \cos 60t - \frac{3}{10} j \sin 60t - \frac{1}{10} \sin 60t \right] + C e^{-20t} \\
I(t) &= -\left[\left[\frac{3}{10} + \frac{j}{10} + \frac{3}{10} - \frac{j}{10} \right] [\cos 60t] + \left[\frac{3j}{10} - \frac{1}{10} - \frac{3j}{10} - \frac{1}{10} \right] [\sin 60t] \right] + C e^{-20t} \\
I(t) &= -\left[\left[\frac{3}{10} + \frac{3}{10} \right] [\cos 60t] - \left[\frac{1}{10} + \frac{1}{10} \right] [\sin 60t] \right] + C e^{-20t} \\
I(t) &= -\left[\left[\frac{6}{10} \right] [\cos 60t] - \left[\frac{2}{10} \right] [\sin 60t] \right] + C e^{-20t} \\
I(t) &= -\left[\left[\frac{3}{5} \right] [\cos 60t] - \left[\frac{1}{5} \right] [\sin 60t] \right] + C e^{-20t} \\
I(t) &= -\left[\frac{3}{5} [\cos 60t] - \frac{1}{5} [\sin 60t] \right] + C e^{-20t} \\
I(t) &= -\frac{3}{5} [\cos 60t] + \frac{1}{5} [\sin 60t] + C e^{-20t} \\
I(t) &= -\frac{3}{5} \cos 60t + \frac{1}{5} \sin 60t + C e^{-20t}
\end{aligned}$$

(a) Find $I(t)$:

$$I(t) = \frac{1}{5} \sin(60t) - \frac{3}{5} \cos(60t) + C e^{-20t}$$

$$I(0) = 1A$$

$$t = 0$$

$$I(t) = \frac{1}{5} \sin(60t) - \frac{3}{5} \cos(60t) + C e^{-20t}$$

$$1 = \frac{1}{5} \sin(60 \times 0) - \frac{3}{5} \cos(60 \times 0) + C e^{-20 \times 0}$$

$$1 = \frac{1}{5} \sin(0) - \frac{3}{5} \cos(0) + C e^0$$

$$1 = 0 - \frac{3}{5} \times 1 + C \times 1$$

$$1 = -\frac{3}{5} + C$$

$$C = 1 + \frac{3}{5} = \frac{8}{5}$$

$$C = \frac{8}{5}$$

$$I(t) = \frac{1}{5}\sin(60t) - \frac{3}{5}\cos(60t) + Ce^{-20t}$$

$$I(t) = \frac{1}{5}\sin(60t) - \frac{3}{5}\cos(60t) + \frac{8}{5}e^{-20t}$$

(b) Find the current after 0.1 seconds.

$t = 0.1$ seconds

$$I(t) = \frac{1}{5}\sin(60t) - \frac{3}{5}\cos(60t) + \frac{8}{5}e^{-20t}$$

$$I(0.1) = \frac{1}{5}\sin(60 \times 0.1) - \frac{3}{5}\cos(60 \times 0.1) + \frac{8}{5}e^{-20 \times 0.1}$$

$$I(0.1) = \frac{1}{5}\sin(6) - \frac{3}{5}\cos(6) + \frac{8}{5}e^{-2}$$

$$I(0.1) = -0.3592709914A$$

$$I(0.1) = -0.36A$$

4. Consider the differential equation

$$t^2 + 2t - 1 - 4yy' = 0$$

Answer the following questions.

(a) What is the order of this equation?

First Order

(b) Is this equation linear?

Non linear

(c) Is this equation separable?

$$t^2 + 2t - 1 - 4yy' = 0$$

$$t^2 + 2t - 1 - 4y \frac{dy}{dt} = 0$$

$$t^2 dt + 2t dt - 1 dt - 4y dy = 0$$

$$[t^2 + 2t - 1]dt - 4y dy = 0$$

$$[t^2 + 2t - 1]dt = 4y dy$$

$$[t^2 + 2t - 1]dt = 4y dy$$

Yes, separable.

(d) Is this equation exact?

$$[t^2 + 2t - 1]dt = 4y dy$$

$$\nabla = \begin{vmatrix} [t^2 + 2t - 1] & 4y \\ dt & dy \end{vmatrix}$$

$$[t^2 + 2t - 1]'_y - [4y]'_t = 0$$

$$[t^2 + 2t - 1]'_y = [4y]'_t$$

$$[0] = [0]$$

Yes, Exact.

(e) Find its general solution.

$$t^2 + 2t - 1 - 4yy' = 0$$

$$[t^2 + 2t - 1]dt = 4ydy$$

$$\int [t^2 + 2t - 1]dt = \int 4ydy$$

$$\left[\frac{t^3}{3} + \frac{2t^2}{2} - t + C \right] = \left[\frac{4y^2}{2} \right]$$

$$\left[\frac{t^3}{3} + t^2 - t + C \right] = [2y^2 + C]$$

$$\frac{t^3}{3} + t^2 - t + C = 2y^2 + C$$

$$\frac{t^3}{3} + t^2 - t = 2y^2$$

$$y^2 = \frac{\frac{t^3}{3} + t^2 - t}{2}$$

$$y = \sqrt{\frac{\frac{t^3}{3} + t^2 - t}{2}}$$

5. Find the general solution of the following second order no
– homogeneous linear

differential equation:

$$y'' - 5y' + 6y = e^{3t} + t^2 - 4$$

$$[D^2 - 5D + 6]y = e^{3t} + t^2 - 4$$

$$[D^2 - 5D + 6]y = 0$$

$$y = 0$$

$$[D^2 - 5D + 6] = 0$$

$$D^2 - 5D + 6 = 0$$

$$\nabla = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{5 \pm \sqrt{5^2 - 4 \times 6}}{2}$$

$$[D - 3][D - 2] = 0$$

$$D_1 = 3$$

$$D_2 = 2$$

$$y_3 = Ae^{3t} + Be^{2t}$$

$$y_3 = e^{3t} + t^2 - 4$$

Y particular Table:

$$y_1 = e^{3t} = Ae^{3t}$$

$$y'_1 = 3Ae^{3t}$$

$$y''_1 = 9Ae^{3t}$$

$$y'' - 5y' + 6y = e^{3t} + t^2 - 4$$

$$9Ae^{3t} - 5[3Ae^{3t}] + 6[Ae^{3t}] = e^{3t}$$

$$9Ae^{3t} - 15Ae^{3t} + 6Ae^{3t} = e^{3t}$$

$$[9 - 15 + 6]Ae^{3t} = e^{3t}$$

$$[0]Ae^{3t} = e^{3t}$$

$$[0]A = 1$$

$$A = \frac{1}{0} = \infty$$

$$A=t$$

$$y_1 = Ae^{3t} = te^{3t}$$

$$y_2 = t^2 - 4 = At^2 + Bt + C$$

$$y_2' = 2At + B$$

$$y_2'' = 2A$$

$$y'' - 5y' + 6y = e^{3t} + t^2 - 4$$

$$2A - 5[2At + B] + 6[At^2 + Bt + C] = t^2 - 4$$

$$2A - 10At - 5B + 6At^2 + 6Bt + 6C = t^2 - 4$$

$$6At^2 = t^2 \quad 6A = 1 \quad A = \frac{1}{6}$$

$$-10At + 6Bt = 0 \quad -10A + 6B = 0 \quad -10\left[\frac{1}{6}\right] + 6B = 0 \quad 6B = 10\left[\frac{1}{6}\right] =$$

$$\frac{5}{3} \quad B = \frac{\frac{5}{3}}{6} = \frac{5}{18}$$

$$2A - 5B + 6C = -4 \quad 2\left[\frac{1}{6}\right] - 5\left[\frac{5}{18}\right] + 6C = -4 \quad \frac{1}{3} - \frac{25}{18} + 6C = -4 \quad 6C =$$

$$-4 - \frac{1}{3} + \frac{25}{18}$$

$$6C = -\frac{53}{18} \quad C = -\frac{\frac{53}{18}}{6} = -\frac{53}{108}$$

$$A = \frac{1}{6} \quad B = \frac{5}{18} \quad C = -\frac{53}{108}$$

$$y_2 = \frac{1}{6}t^2 + \frac{5}{18}t - \frac{53}{108}$$

$$y = y_1 + y_2 + y_3$$

$$y = Ae^{3t} + Be^{2t} + te^{3t} + \frac{1}{6}t^2 + \frac{5}{18}t - \frac{53}{108}$$

6. In the circuit shown in figure 2: $E = 500 \sin 10t$, $R_1 = 10 \text{ Ohms}$, $R_2 = 10 \text{ Ohms}$, $L = 1H$,

$C = 0.01F$. If the currents I_1 and I_2 are zero at $t = 0$, find the laplace transform expression

the current through the inductor L .

$$\begin{aligned} I_1 - I_2 &= I_L \\ E &= Vr_1 + Vc_1 + VL_1 \\ 0 &= Vr_2 - VL_1 \end{aligned}$$

$$\begin{aligned} E &= Vr_1 + Vc_1 + VL_1 \\ E &= I_1(t)r_1 + \frac{1}{C} \int I_1(t) dt + L_1 \frac{dI_L}{dx} \end{aligned}$$

$$\begin{aligned}
E &= I_1(t)r_1 + \frac{1}{C} \int I_1(t) dt + L_1 \frac{d[I_1(t) - I_2(t)]}{dx} \\
500 \sin 10t &= 10I_1(t) + \frac{1}{0.01} \int I_1(t) dt + 1 \frac{d[I_1(t) - I_2(t)]}{dx} \\
500 \sin 10t &= 10I_1(t) + 100 \int I_1(t) dt + \frac{d}{dx} [I_1(t) - I_2(t)] \\
L\{500 \sin 10t\} &= L\left\{10I_1(t) + 100 \int I_1(t) dt + \frac{d}{dx} [I_1(t) - I_2(t)]\right\} \\
500L\{\sin 10t\} &= 10L\{I_1(t)\} + 100L\left\{\int I_1(t) dt\right\} + L\left\{\frac{d}{dx} [I_1(t) - I_2(t)]\right\} \\
500 \left[\frac{10}{s^2 + 100} \right] &= 10I_1(s) + \frac{100}{s} I_1(s) + s[I_1(s) - I_2(s)] \\
500 \left[\frac{10}{s^2 + 100} \right] &= 10I_1(s) + \frac{100}{s} I_1(s) + s[I_1(s) - I_2(s)] \\
500 \left[\frac{10s}{s^2 + 100} \right] &= 10sI_1(s) + 100I_1(s) + s^2[I_1(s) - I_2(s)] \\
500 \left[\frac{10s}{s^2 + 100} \right] &= 10sI_1(s) + 100I_1(s) + s^2I_1(s) - s^2I_2(s) \\
500 \left[\frac{10s}{s^2 + 100} \right] &= [10s + 100 + s^2]I_1(s) - s^2I_2(s) \\
\left[\frac{5000s}{s^2 + 100} \right] &= [s^2 + 100 + 10s]I_1(s) - s^2I_2(s)
\end{aligned}$$

$$\begin{aligned}
0 &= Vr_2 - V_{L_1} \\
0 &= I_2(t)r_2 - L_1 \frac{dI_L}{dx} \\
0 &= I_2(t)r_2 - L_1 \frac{d[I_1(t) - I_2(t)]}{dx} \\
0 &= 10I_2(t) - 1 \frac{d}{dx} [I_1(t) - I_2(t)] \\
0 &= 10I_2(t) - \frac{d}{dx} [I_1(t) - I_2(t)] \\
L\{0\} &= L\left\{10I_2(t) - \frac{d}{dx} [I_1(t) - I_2(t)]\right\} \\
L\{0\} &= 10L\{I_2(t)\} - L\left\{\frac{d}{dx} [I_1(t) - I_2(t)]\right\} \\
0 &= 10I_2(s) - s[I_1(s) - I_2(s)] \\
0 &= 10I_2(s) - sI_1(s) + sI_2(s) \\
0 &= [10 + s]I_2(s) - sI_1(s) \\
0 &= sI_1(s) - [10 + s]I_2(s)
\end{aligned}$$

$$\begin{aligned}
\left[\frac{5000s}{s^2 + 100} \right] &= [s^2 + 100 + 10s]I_1(s) - s^2I_2(s) \\
0 &= sI_1(s) - [10 + s]I_2(s) \\
\nabla &= \begin{vmatrix} [s^2 + 100 + 10s] & -s^2 \\ s & -[10 + s] \end{vmatrix} \\
\nabla &= \{-[s^2 + 100 + 10s] \times [10 + s]\} - \{[s] \times [-s^2]\} \\
\nabla &= -[10s^2 + s^3 + 1000 + 100s + 100s + 10s^2] - [-s^3]
\end{aligned}$$

$$\begin{aligned}
\nabla &= -[20s^2 + s^3 + 1000 + 200s] + s^3 \\
&= -20s^2 - s^3 - 1000 - 200s + s^3 \\
&= -20s^2 - 1000 - 200s \\
\nabla &= -20 \times [s^2 + 10s + 50]
\end{aligned}$$

$$\begin{aligned}
\nabla_{i_1} &= \begin{vmatrix} \left[\frac{5000s}{s^2 + 100} \right] & -s^2 \\ 0 & -[10 + s] \end{vmatrix} \\
\nabla_{i_1} &= \left\{ -\left[\frac{5000s}{s^2 + 100} \right] \times [10 + s] \right\} - \{ [0] \times [-s^2] \} = \left\{ -\left[\frac{5000s}{s^2 + 100} \right] \times [10 + s] \right\} \\
&= -\left[\frac{5000s[10 + s]}{s^2 + 100} \right] = -20 \times \left[\frac{250s[10 + s]}{s^2 + 100} \right] \\
\nabla_{i_1} &= -20 \times \left[\frac{250s[10 + s]}{s^2 + 100} \right]
\end{aligned}$$

$$\begin{aligned}
\nabla_{i_2} &= \begin{vmatrix} [s^2 + 100 + 10s] & \left[\frac{5000s}{s^2 + 100} \right] \\ s & 0 \end{vmatrix} \\
\nabla_{i_2} &= \{ [s^2 + 100 + 10s] \times [0] \} - \left\{ [s] \times \left[\frac{5000s}{s^2 + 100} \right] \right\} = -\left\{ [s] \times \left[\frac{5000s}{s^2 + 100} \right] \right\} \\
&= -20 \times \left[\frac{250s^2}{s^2 + 100} \right] \\
\nabla_{i_2} &= -20 \times \left[\frac{250s^2}{s^2 + 100} \right]
\end{aligned}$$

$$\begin{aligned}
I_1(s) &= \frac{\nabla_{i_1}}{\nabla} = \frac{-20 \times \left[\frac{250s[10 + s]}{s^2 + 100} \right]}{-20 \times [s^2 + 10s + 50]} = \frac{\left[\frac{250s[10 + s]}{s^2 + 100} \right]}{[s^2 + 10s + 50]} = \frac{250s[10 + s]}{[s^2 + 10s + 50][s^2 + 100]} \\
I_2(s) &= \frac{\nabla_{i_2}}{\nabla} = \frac{-20 \times \left[\frac{250s^2}{s^2 + 100} \right]}{-20 \times [s^2 + 10s + 50]} = \frac{\left[\frac{250s^2}{s^2 + 100} \right]}{[s^2 + 10s + 50]} = \frac{250s^2}{[s^2 + 10s + 50][s^2 + 100]}
\end{aligned}$$

$$\begin{aligned}
I_1(s) &= \frac{250s[10 + s]}{[s^2 + 10s + 50][s^2 + 100]} = \frac{250s^2 + 2500s}{[s^2 + 10s + 50][s^2 + 100]} \\
I_1(s) &= \frac{250s^2}{[s^2 + 10s + 50][s^2 + 100]} + \frac{2500s}{[s^2 + 10s + 50][s^2 + 100]} \\
I_2(s) &= \frac{250s^2}{[s^2 + 10s + 50][s^2 + 100]}
\end{aligned}$$

$$\begin{aligned}
I_L(s) &= \frac{I_1(s) - I_2(s)}{250s^2} + \frac{2500s}{[s^2 + 10s + 50][s^2 + 100]} \\
&\quad - \frac{250s^2}{[s^2 + 10s + 50][s^2 + 100]}
\end{aligned}$$

$$I_L(s) = \frac{2500s}{[s^2 + 10s + 50][s^2 + 100]}$$

Partial fraction of $I_L(s)$

$$I_L(s) = \frac{2500s}{[s^2 + 10s + 50][s^2 + 100]}$$

$$\begin{aligned} [s^2 + 10s + 50] = 0 \quad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} &= \frac{-10 \pm \sqrt{10^2 - 4 \times 50}}{2} = \frac{-5 \pm 5j}{1} \\ a^2 + b^2 &= (a + bj)(a - bj) \quad a^2 + b^2 = (a + b)^2 - 2ab \quad a^2 + b^2 \\ &= (a - b)^2 + 2ab \\ [s + 5 \pm 5j] &= [s + 5(1 \pm j)] = [(s + 5) \pm 5j] = [s^2 + 10s + 50] = [s^2 + 10(s + 5)] \\ &= [(s + 10)s + 50] = [s + 5 - 5j][s + 5 + 5j] = [s + 5(1 - j)][s + 5(1 + j)] \\ &= [(s + 5) - (5j)][(s + 5) + (5j)] \\ &= [(s + 5)(s + 5) + (s + 5)(5j) - (s + 5)(5j) - (5j)(5j)] \\ &= [(s + 5)^2 - (5j)^2] = [(s + 5)^2 + 25] = [(s + 5)^2 + (5)^2] \end{aligned}$$

$$\begin{aligned} [s^2 + 100] &= [s^2 + 0s + 100] = 0 \quad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{0 \pm \sqrt{0 - 4 \times 100}}{2} \\ &= \pm 10j \\ a^2 + b^2 &= (a + bj)(a - bj) \quad a^2 + b^2 = (a + b)^2 - 2ab \quad a^2 + b^2 \\ &= (a - b)^2 + 2ab \\ [s \pm 10j] &= [s^2 + 100] = [s^2 + 0s + 100] = [s + 10j][s - 10j] \\ &= [(s) + (10j)][(s) - (10j)] \\ &= [(s)(s) + (s)(10j) - (s)(10j) - (10j)(10j)] = [(s)^2 - (10j)^2] = [s^2 + 100] \\ &= [s^2 + 10^2] \end{aligned}$$

$$\begin{aligned} I_L(s) &= \frac{2500s}{[s^2 + 10s + 50][s^2 + 100]} = \frac{As + B}{[s^2 + 10s + 50]} + \frac{Cs + D}{[s^2 + 100]} = P_1 + P_2 \\ &= \frac{As + B}{[(s + 5)^2 + (5)^2]} + \frac{Cs + D}{[s^2 + 10^2]} = \frac{As + B}{[s + 5 - 5j][s + 5 + 5j]} + \frac{Cs + D}{[s + 10j][s - 10j]} \\ &= \frac{A}{[s + 5 - 5j]} + \frac{B}{[s + 5 + 5j]} + \frac{C}{[s + 10j]} + \frac{D}{[s - 10j]} \end{aligned}$$

Partial fraction of $[s^2 + 10s + 50]$

$$\begin{aligned} P_1 &= \frac{2500s}{[s^2 + 10s + 50]} = \frac{As + B}{[(s + 5)^2 + (5)^2]} = \frac{As + B}{[s + 5 - 5j][s + 5 + 5j]} \\ &= \frac{A}{[s + 5 - 5j]} + \frac{B}{[s + 5 + 5j]} \\ 2500s &= A[s + 5 + 5j] + B[s + 5 - 5j] \\ s = -5 - 5j \quad \text{we get} \quad 2500[-5 - 5j] &= A[(-5 - 5j) + 5 + 5j] + B[(-5 - 5j) + 5 - 5j] \\ s = -5 - 5j \quad \text{we get} \quad 2500[-5 - 5j] &= A[(-5 - 5j) + 5 + 5j] + B[(-5 - 5j) + 5 - 5j] \\ 2500[-5 - 5j] &= A[0] + B[-5 - 5j + 5 - 5j] \\ 2500[-5 - 5j] &= B[-5j - 5j] \\ 2500[-5 - 5j] &= B[-10j] \end{aligned}$$

$$\begin{aligned}
 -12500[1+j] &= B[-10j] \\
 1250[1+j] &= B[j] \\
 B &= 1250 \times \frac{[1+j]}{[j]} = 1250 \times \frac{j[1+j]}{[j]^2} = -1250 \times j[1+j] = -1250 \times [j-1] \\
 B &= -1250[j-1]
 \end{aligned}$$

$$\begin{aligned}
 s = -5 + 5j \quad \text{we get} \quad 2500[-5 + 5j] &= A[(-5 + 5j) + 5 + 5j] + B[(-5 + 5j) + 5 - 5j] \\
 2500[-5 + 5j] &= A[-5 + 5j + 5 + 5j] + B[-5 + 5j + 5 - 5j] \\
 2500[-5 + 5j] &= A[5j + 5j] + B[0] \\
 12500[-1 + j] &= A[10j] \\
 1250[-1 + j] &= A[j] \\
 A &= 1250 \times \frac{[-1 + j]}{[j]} = 1250 \times \frac{j[-1 + j]}{[j]^2} = -1250 \times j[-1 + j] = -1250 \times [-1 - j] \\
 &= 1250 \times [j + 1] \\
 A &= 1250[j + 1]
 \end{aligned}$$

$$\begin{aligned}
 P_1 &= \frac{\frac{A = 1250[j + 1]}{2500s}}{[s^2 + 10s + 50]} = \frac{\frac{As + B}{As + B}}{[(s + 5)^2 + (5)^2]} = \frac{\frac{B = -1250[j - 1]}{As + B}}{[s + 5 - 5j][s + 5 + 5j]} \\
 &= \frac{A}{[s + 5 - 5j]} + \frac{B}{[s + 5 + 5j]} \\
 P_1 &= \frac{1250[j + 1]}{[s + 5 - 5j]} - \frac{1250[j - 1]}{[s + 5 + 5j]}
 \end{aligned}$$

$$\begin{aligned}
 P_2 &= \frac{2500s}{[s^2 + 100]} = \frac{\text{Partial fraction of } [s^2 + 100]}{[s^2 + 10^2]} = \frac{Cs + D}{[s + 10j][s - 10j]} = \frac{C}{[s + 10j]} + \frac{D}{[s - 10j]} \\
 2500s &= C[s - 10j] + D[s + 10j] \\
 s = -10j \quad \text{we get} \quad 2500s &= C[(-10j) - 10j] + D[(-10j) + 10j] \\
 2500s &= C[-10j - 10j] + D[-10j + 10j] \\
 2500s &= C[-20j] + D[0] \\
 2500s &= C[-20j] \\
 -125s &= C[j] \\
 C &= \frac{-125}{[j]} = \frac{-125[j]}{[j]^2} = 125j \\
 C &= 125j
 \end{aligned}$$

$$\begin{aligned}
 s = 10j \quad \text{we get} \quad 2500s &= C[(10j) - 10j] + D[(10j) + 10j] \\
 2500s &= C[10j - 10j] + D[10j + 10j] \\
 2500s &= C[0] + D[20j] \\
 2500s &= D[20j] \\
 125s &= D[j] \\
 D &= \frac{125}{[j]} = \frac{125[j]}{[j]^2} = -125j \\
 D &= -125j
 \end{aligned}$$

$$P_2 = \frac{2500s}{[s^2 + 100]} = \frac{Cs + D}{[s^2 + 10^2]} = \frac{Cs + D}{[s + 10j][s - 10j]} = \frac{C}{[s + 10j]} + \frac{D}{[s - 10j]}$$

$$P_2 = \frac{125j}{[s + 10j]} - \frac{125j}{[s - 10j]}$$

Partial fraction of $I_L(s)$

$$I_L(s) = \frac{2500s}{[s^2 + 10s + 50][s^2 + 100]} = P_1 + P_2$$

$$I_L(s) = \frac{1250[j + 1]}{[s + 5 - 5j]} - \frac{1250[j - 1]}{[s + 5 + 5j]} + \frac{125j}{[s + 10j]} - \frac{125j}{[s - 10j]}$$

$$e^{bt} = \frac{1}{s - b} \quad e^{bt} \sin at = \frac{a}{(s - b)^2 + a^2} \quad e^{bt} \cos at = \frac{s - b}{(s - b)^2 + a^2}$$

$$I_L(t) = \text{inverse laplace} \left\{ \frac{\text{inverse laplace of } I_L(s)}{\left[\frac{1250[j + 1]}{[s + (5 - 5j)]} - \frac{1250[j - 1]}{[s + (5 + 5j)]} + \frac{125j}{[s + (10j)]} - \frac{125j}{[s - (10j)]} \right]} \right\}$$

$$\text{inverse laplace of } \frac{1250[j + 1]}{[s + (5 - 5j)]} = 1250[j + 1]e^{-(5-5j)t}$$

$$\text{inverse laplace of } -\frac{1250[j - 1]}{[s + (5 + 5j)]} = -1250[j - 1]e^{-(5+5j)t}$$

$$\text{inverse laplace of } \frac{125j}{[s + (10j)]} = 125j e^{-(10j)t}$$

$$\text{inverse laplace of } -\frac{125j}{[s - (10j)]} = -125j e^{(10j)t}$$

$$I_L(t) = 1250[j + 1]e^{-(5-5j)t} - 1250[j - 1]e^{-(5+5j)t} + 125j e^{-(10j)t} - 125j e^{(10j)t}$$

$$I_L(t) = 1250[j + 1][e^{-(5-5j)t}] - 1250[j - 1][e^{-(5+5j)t}] + 125j [e^{-(10j)t}] - 125j [e^{(10j)t}]$$

$$I_L(t) = 1250[j + 1][e^{-5t+5jt}] - 1250[j - 1][e^{-5t-5jt}] + 125j [e^{-10jt}] - 125j [e^{10jt}]$$

$$I_L(t) = 1250[j + 1][e^{-5t}e^{5jt}] - 1250[j - 1][e^{-5t}e^{-5jt}] + 125j [e^{-10jt}] - 125j [e^{10jt}]$$

$$[j + 1] = 2 < 45^\circ = 2 < \frac{\pi}{4} = 2e^{j45^\circ t} = 2e^{j\frac{\pi}{4}t}$$

$$a = \sqrt{a^2 + b^2} = \sqrt{1^2 + 1^2} = 2 \quad \theta = \tan^{-1} \left[\frac{1}{1} \right] = 45^\circ$$

$$\pi \rightarrow 180^\circ \quad 1^\circ \rightarrow \frac{\pi}{180^\circ} \quad 45^\circ \rightarrow 45^\circ \times \frac{\pi}{180^\circ} = \frac{\pi}{4}$$

$$[j - 1] = 2 < 45^\circ = 2 < -\frac{\pi}{4} = 2e^{-j45^\circ t} = 2e^{-j\frac{\pi}{4}t}$$

$$125j = 0 + 125j = 125 < 90^\circ = 125 < \frac{\pi}{2} = 125e^{j90^\circ t} = 125e^{j\frac{\pi}{2}t}$$

$$a = \sqrt{a^2 + b^2} = \sqrt{0^2 + 125^2} = 125$$

$$= \frac{\pi}{2}$$

$$\pi \rightarrow 90^\circ$$

$$1^\circ \rightarrow \frac{\pi}{180^\circ}$$

$$\theta = \tan^{-1} \left[\frac{125}{0} \right] = \tan^{-1}[\infty] = 90^\circ$$

$$90^\circ \rightarrow 90^\circ \times \frac{\pi}{180^\circ} = \frac{\pi}{2}$$

$$-125j = 0 - 125j = 125 < -90^\circ = 125 < -\frac{\pi}{2} = 125e^{-j90^\circ t} = 125e^{-j\frac{\pi}{2}t}$$

$$I_L(t) = 1250[j + 1][e^{-5t}e^{5jt}] - 1250[j - 1][e^{-5t}e^{-5jt}] + 125j[e^{-10jt}] - 125j[e^{10jt}]$$

$$I_L(t) = 1250 \left[2e^{j\frac{\pi}{4}t} \right] [e^{-5t}e^{5jt}] - 1250 \left[2e^{-j\frac{\pi}{4}t} \right] [e^{-5t}e^{-5jt}] + 125e^{j\frac{\pi}{2}t} [e^{-10jt}]$$

$$- 125e^{-j\frac{\pi}{2}t} [e^{10jt}]$$

$$I_L(t) = 2500 \left[e^{j\frac{\pi}{4}t} \right] [e^{-5t}e^{5jt}] - 2500 \left[e^{-j\frac{\pi}{4}t} \right] [e^{-5t}e^{-5jt}] + 125e^{j\frac{\pi}{2}t} [e^{-10jt}]$$

$$- 125e^{-j\frac{\pi}{2}t} [e^{10jt}]$$

$$I_L(t) = 2500[e^{-5t}] \left[e^{j\frac{\pi}{4}t}e^{5jt} \right] - 2500[e^{-5t}] \left[e^{-j\frac{\pi}{4}t}e^{-5jt} \right] + 125 \left[e^{j\frac{\pi}{2}t}e^{-10jt} \right]$$

$$- 125 \left[e^{-j\frac{\pi}{2}t}e^{10jt} \right]$$

$$I_L(t) = 2500[e^{-5t}] \left[e^{j\frac{\pi}{4}t}e^{5jt} \right] - 2500[e^{-5t}] \left[e^{-j\frac{\pi}{4}t}e^{-5jt} \right] + 125 \left[e^{j\frac{\pi}{2}t}e^{-10jt} \right]$$

$$- 125 \left[e^{-j\frac{\pi}{2}t}e^{10jt} \right]$$

$$I_L(t) = 2500[e^{-5t}] \left[e^{5jt+j\frac{\pi}{4}t} \right] - 2500[e^{-5t}] \left[e^{-5jt-j\frac{\pi}{4}t} \right] + 125 \left[e^{-10jt+j\frac{\pi}{2}t} \right]$$

$$- 125 \left[e^{10jt-j\frac{\pi}{2}t} \right]$$

$$= 2500[e^{-5t}] \left[e^{(5jt+j\frac{\pi}{4}t)} \right] - 2500[e^{-5t}] \left[e^{(-5jt-j\frac{\pi}{4}t)} \right] + 125 \left[e^{(-10jt+j\frac{\pi}{2}t)} \right]$$

$$- 125 \left[e^{(10jt-j\frac{\pi}{2}t)} \right]$$

$$= 2500[e^{-5t}] \left[e^{j(5+\frac{\pi}{4})t} \right] - 2500[e^{-5t}] \left[e^{-j(5+\frac{\pi}{4})t} \right] + 125 \left[e^{-j(10-\frac{\pi}{2})t} \right] - 125 \left[e^{j(10-\frac{\pi}{2})t} \right]$$

$$= 2500[e^{-5t}] \left\{ \left[e^{j(5+\frac{\pi}{4})t} \right] - \left[e^{-j(5+\frac{\pi}{4})t} \right] \right\} + 125 \left\{ \left[e^{-j(10-\frac{\pi}{2})t} \right] - \left[e^{j(10-\frac{\pi}{2})t} \right] \right\}$$

$$= 2500[e^{-5t}] \left[e^{j(5+\frac{\pi}{4})t} - e^{-j(5+\frac{\pi}{4})t} \right] + 125 \left[e^{-j(10-\frac{\pi}{2})t} - e^{j(10-\frac{\pi}{2})t} \right]$$

$$= 2500[e^{-5t}] \left[e^{j(5+\frac{\pi}{4})t} - e^{-j(5+\frac{\pi}{4})t} \right] - 125 \left[e^{j(10-\frac{\pi}{2})t} - e^{-j(10-\frac{\pi}{2})t} \right]$$

Euler formula:

$$\begin{cases} e^{jat} = \cos at + j \sin at & [1] \\ e^{-jat} = \cos at - j \sin at & [2] \end{cases}$$

$$e^{jat} - e^{-jat} = \cos at + j \sin at - \cos at + j \sin at \quad [1] - [2]$$

$$e^{jat} - e^{-jat} = j \sin at + j \sin at$$

$$e^{jat} - e^{-jat} = j2 \sin at$$

$$j2 \sin at = e^{jat} - e^{-jat}$$

$$\sin at = \frac{e^{jat} - e^{-jat}}{j2}$$

$$\left[e^{j(5+\frac{\pi}{4})t} - e^{-j(5+\frac{\pi}{4})t} \right] = j2 \sin \left(5 + \frac{\pi}{4} \right) t$$

$$\left[e^{j\left(10-\frac{\pi}{2}\right)t} - e^{-j\left(10-\frac{\pi}{2}\right)t} \right] = j2 \sin\left(10 - \frac{\pi}{2}\right)t$$

$$I_L(t) = 2500[e^{-5t}] \left[e^{j\left(5+\frac{\pi}{4}\right)t} - e^{-j\left(5+\frac{\pi}{4}\right)t} \right] - 125 \left[e^{j\left(10-\frac{\pi}{2}\right)t} - e^{-j\left(10-\frac{\pi}{2}\right)t} \right]$$

$$I_L(t) = 2500[e^{-5t}] \left[j2 \sin\left(5 + \frac{\pi}{4}\right)t \right] - 125 \left[j2 \sin\left(10 - \frac{\pi}{2}\right)t \right]$$

$$I_L(t) = j5000[e^{-5t}] \left[\sin\left(5 + \frac{\pi}{4}\right)t \right] - j250 \left[\sin\left(10 - \frac{\pi}{2}\right)t \right]$$

Trigonometric identities

$$\left[\sin\left(5 + \frac{\pi}{4}\right)t \right] = [\sin(5^\circ + 45^\circ)t] = [\sin(50^\circ)t] = \sin(50^\circ)t$$

$$\left[\sin\left(10 - \frac{\pi}{2}\right)t \right] = [\sin(10^\circ - 90^\circ)t] = [\sin(-80^\circ)t] = -[\sin(80^\circ)t] = -\sin(80^\circ)t$$

$$I_L(t) = j5000[e^{-5t}][\sin(50^\circ)t] - j250 [\sin(-80^\circ)t]$$

$$I_L(t) = j5000[e^{-5t}] \sin(50^\circ)t + j250 \sin(80^\circ)t$$

$$I_L(t) = j250\{20[e^{-5t}] \sin(50^\circ)t + \sin(80^\circ)t\}$$

$$j250 = 0 + j250 = 250 < 90^\circ = 250 < \frac{\pi}{2} = 250e^{i90^\circ t} = 250e^{i\frac{\pi}{2}t}$$

$$I_L(t) = j250\{20[e^{-5t}] \sin(50^\circ)t + \sin(80^\circ)t\}$$

$$I_L(t) = 250[e^{i90^\circ t}]\{20[e^{-5t}] \sin(50^\circ)t + \sin(80^\circ)t\}$$

another way to find the inverse of the partial fraction

$$\begin{aligned} I_L(s) &= \frac{2500s}{[s^2 + 10s + 50][s^2 + 100]} = \frac{As + B}{[s^2 + 10s + 50]} + \frac{Cs + D}{[s^2 + 100]} = P_1 + P_2 \\ &= \frac{As + B}{[(s + 5)^2 + (5)^2]} + \frac{Cs + D}{[s^2 + 10^2]} = \frac{As + B}{[s + 5 - 5j][s + 5 + 5j]} + \frac{Cs + D}{[s + 10j][s - 10j]} \\ &= \frac{A}{[s + 5 - 5j]} + \frac{B}{[s + 5 + 5j]} + \frac{C}{[s + 10j]} + \frac{D}{[s - 10j]} \\ A &= 1250[j + 1] \quad B = -1250[j - 1] \\ C &= 125j \quad D = -125j \\ As + B &= 1250[j + 1]s - 1250[j - 1] \\ Cs + D &= 125js - 125j \end{aligned}$$

$$\begin{aligned} p(t)_1 &= \frac{As + B}{[(s + 5)^2 + (5)^2]} = \frac{1250[j + 1]s - 1250[j - 1]}{[(s + 5)^2 + (5)^2]} \\ &= \frac{1250js + 1250s - 1250j + 1250}{[(s + 5)^2 + (5)^2]} \end{aligned}$$

$$\begin{aligned} &= \frac{1250[j + 1]s}{[(s + 5)^2 + (5)^2]} - \frac{1250[j - 1]}{[(s + 5)^2 + (5)^2]} = \frac{1250[j + 1]s}{[(s + 5)^2 + (5)^2]} - \frac{250[j - 1][5]}{[(s + 5)^2 + (5)^2]} \\ &= \frac{1250js}{[(s + 5)^2 + (5)^2]} + \frac{1250s}{[(s + 5)^2 + (5)^2]} - \frac{1250j}{[(s + 5)^2 + (5)^2]} + \frac{1250}{[(s + 5)^2 + (5)^2]} \end{aligned}$$

one partial term does not match correctly

$$p(s)_1 = 1250[j + 1] \cos 5t - 250[j - 1] e^{-5t} \sin 5t$$

$$\begin{aligned}
Cs + D &= 125js - 125j \\
p(s)_2 &= \frac{Cs + D}{[s^2 + 10^2]} = \frac{125js - 125j}{[s^2 + 10^2]} = \frac{125js}{[s^2 + 10^2]} - \frac{125j}{[s^2 + 10^2]} \\
&= \frac{125js}{[s^2 + 10^2]} - \frac{10 \times 12,5j}{[s^2 + 10^2]} \\
p(t)_1 &= 125j \cos 10t - 12,5j \sin 10t \\
I_L(s) &= \frac{2500s}{[s^2 + 10s + 50][s^2 + 100]} = \frac{As + B}{[(s + 5)^2 + (5)^2]} + \frac{Cs + D}{[s^2 + 10^2]} \\
I_L(s) &= p(t)_1 + p(s)_2 \\
I_L(s) &= 1250[j + 1] \cos 5t - 250[j - 1] e^{-5t} \sin 5t + 125j \cos 10t - 12,5j \sin 10t \\
I_L(s) &= 1250[2e^{j45^\circ t}] \cos 5t - 250[2e^{-j45^\circ t}] e^{-5t} \sin 5t + 125e^{j90^\circ t} \cos 10t \\
&\quad - 12,5e^{j90^\circ t} \sin 10t \\
I_L(s) &= 2500[e^{j45^\circ t}] \cos 5t - 500[e^{-j45^\circ t}] e^{-5t} \sin 5t + 125e^{j90^\circ t} \cos 10t \\
&\quad - 12,5e^{j90^\circ t} \sin 10t
\end{aligned}$$

7. Let $f(t) = t^3$, $0 < t < 2$

(a) Consider the **odd** periodic extension, of period $T=4$, of $f(t)$. Sketch 3 periods, on the interval $[-6,6]$, of this function.

(b) Find the average value of the periodic function described in (a).

$$\begin{aligned}
avg &= \frac{a_0}{2} \quad a_0 = 0 \quad avg = \frac{0}{2} = 0 \\
a_0 &= \frac{2}{4} \int_{-2}^2 t^3 dt = \frac{2}{4} \int_0^2 t^3 dt + \frac{2}{4} \int_{-2}^0 t^3 dt = \frac{1}{2} \int_0^2 t^3 dt + \frac{1}{2} \int_{-2}^0 t^3 dt = \frac{1}{2} [4] + \frac{1}{2} [-4] \\
&= 2 - 2 = 0
\end{aligned}$$

(c) Consider the **Even** periodic extension, of periodic $T=4$, of $f(t)$. Sketch 3 periods, on the interval $[-6,6]$, of this function.

(d) Find the average value of the periodic function described in (c).

$$\begin{aligned}
avg &= \frac{a_0}{2} \\
a_0 &= \frac{2}{4} \int_{-2}^2 t^3 dt = \frac{2}{4} \int_0^2 t^3 dt - \frac{2}{4} \int_{-2}^0 t^3 dt = \frac{1}{2} \int_0^2 t^3 dt - \frac{1}{2} \int_{-2}^0 t^3 dt = \frac{1}{2} [4] - \frac{1}{2} [-4] \\
&= 2 + 2 = 4 \\
avg &= \frac{a_0}{2} = \frac{4}{2} = 2
\end{aligned}$$

(e) State True/False with reason. For the same even periodic extension mentioned in (c), the Fourier sine coefficients are given by

$$\begin{aligned}
b_n &= \int_0^p f(t) \sin \left[\frac{n\pi t}{p} \right] dt = \int_0^2 t^3 \sin \left[\frac{n\pi t}{2} \right] dt \\
b_n &= \frac{2}{p} \int_{-p}^p f(t) \sin \left[\frac{2n\pi t}{p} \right] dt = \frac{2}{4} \int_{-2}^2 t^3 \sin \left[\frac{2n\pi t}{4} \right] dt = \frac{1}{2} \int_{-2}^2 t^3 \sin \left[\frac{n\pi t}{2} \right] dt
\end{aligned}$$

False:

8. Given

$$3(x[k+1])^2 - 2x[k] = k^2$$

$$x[0] = 2$$

Find $x[1], x[2], x[3]$.

$$x[k] = 2$$

$$x[0] = 2$$

$$k = 0$$

$$3(x[k+1])^2 - 2x[k] = k^2$$

$$3(x[0+1])^2 - 2x[0] = [0]^2$$

$$3(x[1])^2 - 2x[0] = 0$$

$$3(x[1])^2 = 2x[0]$$

$$(x[1])^2 = \frac{2x[0]}{3} = \frac{2 \times 2}{3} = \frac{4}{3}$$

$$x[1] = \sqrt{\frac{4}{3}} = 1.1547$$

$$x[k] = 2$$

$$x[1] = \sqrt{\frac{4}{3}} = 1.1547$$

$$x[1] = ?$$

$$k = 1$$

$$3(x[k+1])^2 - 2x[k] = k^2$$

$$3(x[1+1])^2 - 2x[1] = [1]^2$$

$$3(x[2])^2 - 2x[1] = 0$$

$$3(x[2])^2 = 2x[1] = \frac{2 \times \sqrt{\frac{4}{3}}}{3} = \frac{3 + 4\sqrt{3}}{3}$$

$$(x[2])^2 = \frac{2x[1]}{3} = \frac{\frac{3 + 4\sqrt{3}}{3}}{3} = \frac{3 + 4\sqrt{3}}{9}$$

$$x[2] = \sqrt{\frac{3 + 4\sqrt{3}}{9}} = 1.050301$$

$$x[k] = 2$$

$$x[2] = \sqrt{\frac{3 + 4\sqrt{3}}{9}} = 1.050301$$

$$x[2] = ?$$

$$k = 2$$

$$3(x[k+1])^2 - 2x[k] = k^2$$

$$3(x[2+1])^2 - 2x[2] = [2]^2$$

$$3(x[3])^2 - 2x[2] = 4$$

$$3(x[3])^2 = 2x[2] + 4 = 2 \left[\sqrt{\frac{3 + 4\sqrt{3}}{9}} \right] + 4$$

$$(x[3])^2 = \frac{2 \left[\sqrt{\frac{3 + 4\sqrt{3}}{9}} \right] + 4}{3} = 2.03353$$

$$x[3] = \sqrt{2.03353} = 1.42602$$

$$x[k] = 2$$

$$x[2] = 1.42602$$

$$x[3] = ?$$

$$k = 3$$

$$3(x[k+1])^2 - 2x[k] = k^2$$

$$3(x[3+1])^2 - 2x[3] = [3]^2$$

$$3(x[4])^2 - 2x[3] = 9$$

$$3(x[4])^2 = 2x[3] + 9 = 2[1.42602] + 9$$

$$(x[4])^2 = \frac{2[1.42602] + 9}{3}$$

$$x[4] = \sqrt{\frac{2[1.42602] + 9}{3}} = 1.987631841$$

9. Calculate the fourier transform of the windowed sine function:

$$f(t) = \sin 2t [u(t+1) - u(t-1)]$$

$$\sin 2t, \quad -1 < t < 1$$

$$\begin{aligned} \int_{-1}^1 \sin(2t) e^{-j\omega t} dt &= \int_{-1}^1 \left[\frac{e^{j2t} - e^{-j2t}}{2j} \right] e^{-j\omega t} dt = \frac{1}{2j} \int_{-1}^1 [e^{j2t} - e^{-j2t}] e^{-j\omega t} dt \\ &= \frac{1}{2j} \int_{-1}^1 [e^{-j\omega t} e^{j2t} - e^{-j\omega t} e^{-j2t}] dt = \frac{1}{2j} \int_{-1}^1 [e^{-j\omega t + j2t} - e^{-j\omega t - j2t}] dt \\ &= \frac{1}{2j} \int_{-1}^1 [e^{(-j\omega + j2)t} - e^{(-j\omega - j2)t}] dt = \frac{1}{2j} \int_{-1}^1 [e^{j(-\omega + 2)t} - e^{-j(\omega + 2)t}] dt \\ &= \frac{1}{2j} \left[\frac{e^{j(-\omega + 2)t}}{j(-\omega + 2)} - \frac{e^{-j(\omega + 2)t}}{-j(\omega + 2)} \right] = \frac{1}{2j} \left[\frac{e^{j(-\omega + 2)t}}{j(2 - \omega)} + \frac{e^{-j(\omega + 2)t}}{j(\omega + 2)} \right] \\ &= \frac{1}{2j} \left[\frac{e^{j(2 - \omega)} - e^{-j(2 - \omega)}}{j(2 - \omega)} + \frac{e^{-j(\omega + 2)} - e^{j(\omega + 2)}}{j(\omega + 2)} \right] \\ &= \frac{1}{2j} \left[\frac{e^{j(2 - \omega)} - e^{-j(2 - \omega)}}{j(2 - \omega)} - \frac{e^{j(\omega + 2)} - e^{-j(\omega + 2)}}{j(\omega + 2)} \right] \\ &= \frac{1}{j} \left[\frac{e^{j(2 - \omega)} - e^{-j(2 - \omega)}}{j2(2 - \omega)} - \frac{e^{j(\omega + 2)} - e^{-j(\omega + 2)}}{j2(\omega + 2)} \right] = \frac{1}{j} \left[\frac{\sin(2 - \omega)}{(2 - \omega)} - \frac{\sin(\omega + 2)}{(\omega + 2)} \right] \\ &= \frac{1}{j} [\text{sinc}(2 - \omega) - \text{sinc}(\omega + 2)] \end{aligned}$$

$$C_n = -j[\text{sinc}(2 - \omega) - \text{sinc}(\omega + 2)]$$

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Author information: Henock, also known as HNK, is a computer engineering student with a passion for combining technology and design. He seeks simplicity and elegance in his code and strives to deliver intuitive user experiences. With his background in engineering and design, Henock brings a unique perspective and wealth of knowledge to his writing on advanced mathematics.

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