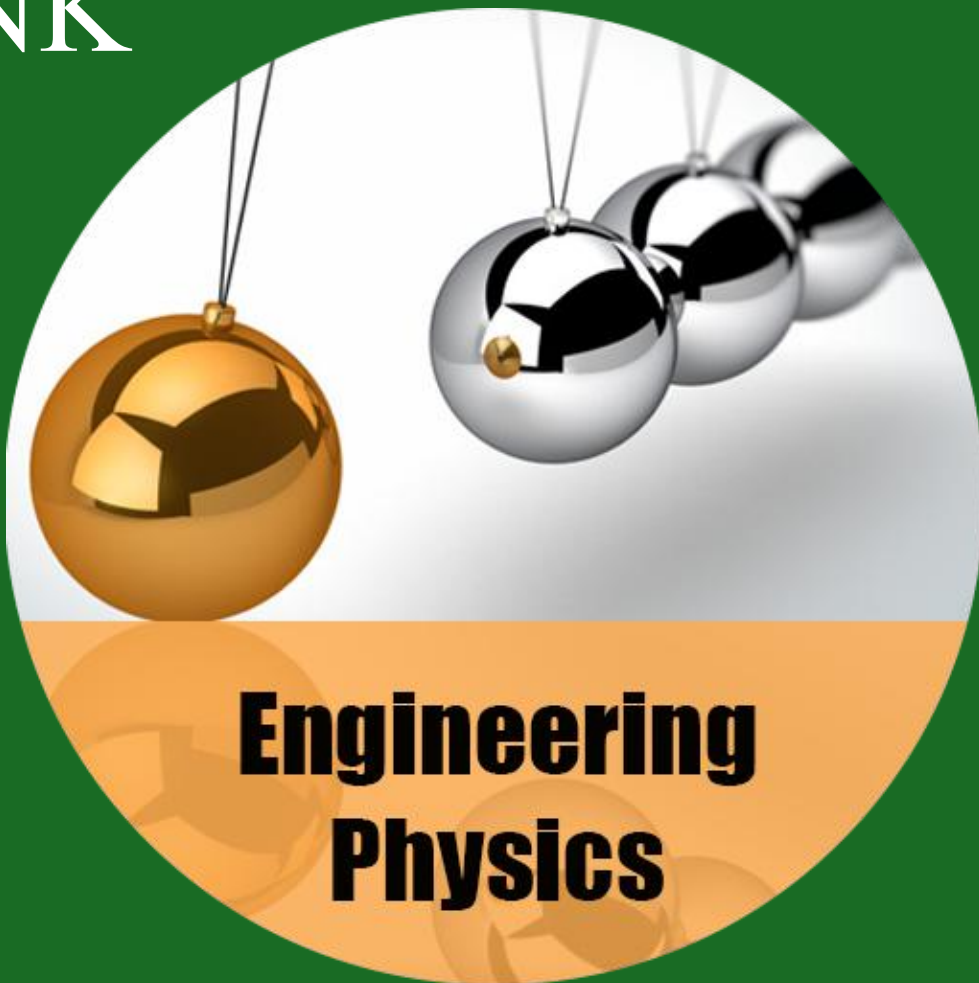


ENGINEERING PHYSICS

B E C O M E R E A D Y



HNK



LEARN Physics 5



LEARN PHYSICS 5

Complete study guide, tests and exam preparation

The original guide is followed by the supplied concept notes, worked examples and exam material. Equations, diagrams and handwritten notes are preserved in their source layouts.

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Calculus and trigonometry	25
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Select a topic above or use the PDF bookmarks.

HOW TO USE THIS GUIDE

1. Start with the original notes

Review the coordinate systems, vector methods and other handwritten material from the original guide.

2. Work through each topic

The added material brings together mathematical foundations, concept explanations and worked test or exam solutions. Alternative methods are kept where supplied.

3. Practise before reading the solution

Cover the worked steps, identify the known values and required quantity, then choose the formula and solve independently.

4. Check the diagrams and units

Use the supplied geometry sketches to define directions, limits and coordinates. Check units and signs at each step.

5. Revise with the formula reference

Use the drawing checklist and formula reference at the end to review the main results.

The two RE-FISA files are overlapping versions. The expanded version supplies the main exam material; selected earlier derivations are retained. Repeated document covers, old contents pages and production notes are omitted.

This guide is designed to help you navigate through the essential topics in **Physics 5**. Each chapter provides a focused understanding of various types of **coordinate conversions**, **vector fields**, and **signal analysis concepts** that will be crucial for exam preparation. The notes outline the key areas, marking their importance based on the percentage they contribute to the exam.

OVERVIEW

Prepare to dive into the essential topics of **Vector Calculus and Dynamic Systems** with this comprehensive guide. Focused on core engineering areas, this resource covers **3D coordinate transformations (Cartesian, Cylindrical, and Spherical)**, **differential vector operators (Gradient, Divergence, and Curl)**, and **signal functions including unit step, impulse, and ramp functions**.

Each chapter is designed to support your learning with targeted insights and rigorous preparation for your upcoming engineering exams. This guide outlines critical mathematical operations and their specific weight in the curriculum, helping you to prioritize your study time effectively for the **Physics 5** module.

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Rectangular

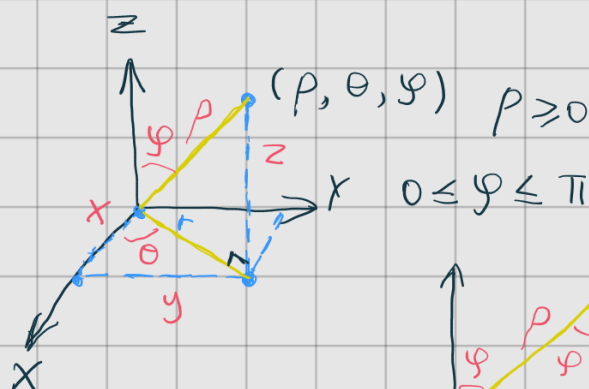


$$\rho = \sqrt{x^2 + y^2 + z^2}$$

$$\tan \theta = \frac{y}{x}$$

$$\phi = \cos^{-1} \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right)$$

Spherical



$$x = r \cos \theta = \rho \sin \phi \cos \theta$$

$$y = r \sin \theta = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \theta$$

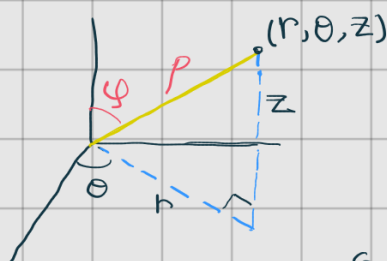


$$\cos \phi = \frac{z}{\rho}$$

$$\sin \phi = \frac{r}{\rho}$$

$$r = \rho \sin \phi$$

Cylindrical



$$\rho = \sqrt{r^2 + z^2}$$

$$\theta = \theta$$

$$\phi = \cos^{-1} \left(\frac{z}{\sqrt{r^2 + z^2}} \right)$$

$$\cos \phi = \frac{z}{\rho}$$

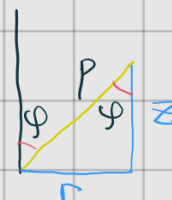
Spherical



$$r = \rho \sin \phi$$

$$\theta = \theta$$

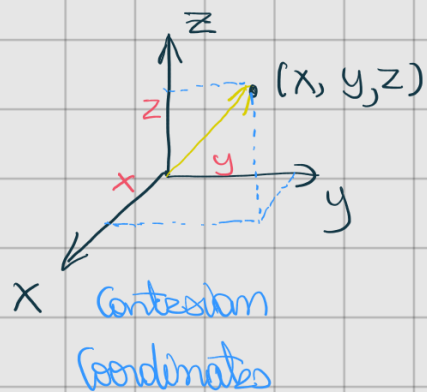
$$z = \rho \cos \phi$$



$$\sin \phi = \frac{r}{\rho}$$

Coordinate Systems

x, y, z, t



Polar Coordinates (r, θ)



$$\tan \theta = \frac{y}{x}$$

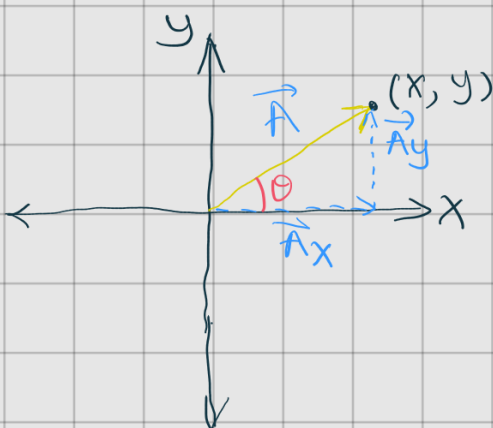
$$r = \sqrt{x^2 + y^2}$$



$$\cos \theta = \frac{x}{r}$$

$$\sin \theta = \frac{y}{r}$$

Vector Components



$$\vec{A} = \vec{A}_x + \vec{A}_y$$

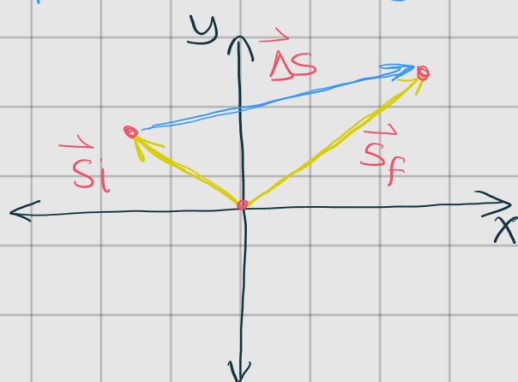
$$A_x = |\vec{A}_x| = A \cos \theta$$

$$A_y = |\vec{A}_y| = A \sin \theta$$

$$A = \sqrt{A_x^2 + A_y^2}$$

$$\theta = \tan^{-1} \left(\frac{A_y}{A_x} \right)$$

Displacement Vector

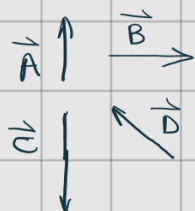


$$\vec{\Delta S} = \vec{S}_f - \vec{S}_i$$

$$\vec{\Delta S} = \vec{S}_f + (-\vec{S}_i)$$

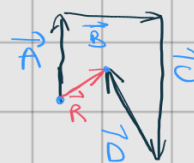


Adding Vectors



Resultant Vector $\vec{R}$

$$\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$$



Vector Laws

Commutative Law

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

Associative Law

$$\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$$

Subtraction

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$

Multiply by scalar

$$\vec{A} \quad \quad \quad 5\vec{A}$$

Unit Vectors

Dimensionless Vector of magnitude = 1

Specify direction

$$x \Rightarrow \hat{i}$$

$$y \Rightarrow \hat{j}$$

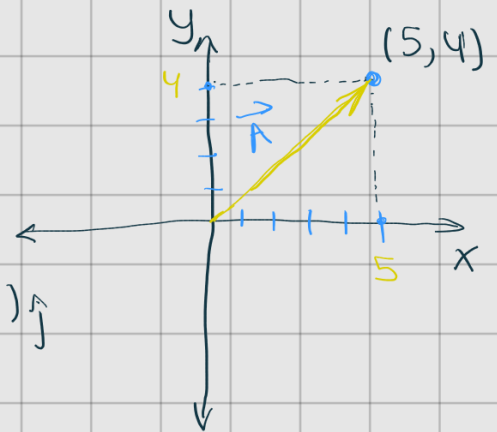
$$z \Rightarrow \hat{k}$$

hat

$\hat{A} \rightarrow$ unit vector

$$\vec{A} = (A_x) \hat{i} + (A_y) \hat{j}$$

$$\vec{A} = 5\hat{i} + 4\hat{j}$$



Adding Vector Components

$$\vec{A} = A_x \hat{i} + A_y \hat{j}$$

$$\vec{B} = B_x \hat{i} + B_y \hat{j}$$

$$\vec{C} = \vec{A} + \vec{B} = (A_x \hat{i} + A_y \hat{j}) + (B_x \hat{i} + B_y \hat{j}) = A_x \hat{i} + B_x \hat{i} + A_y \hat{j} + B_y \hat{j}$$

$$\vec{C} = (A_x + B_x) \hat{i} + (A_y + B_y) \hat{j}$$



$$C_x = A_x + B_x$$

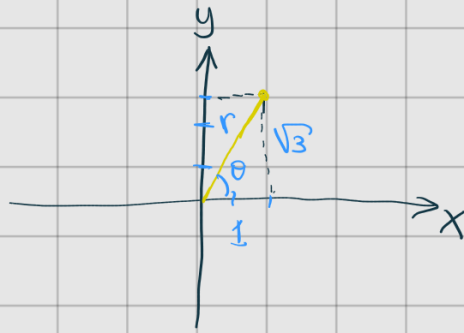
$$C_y = A_y + B_y$$

$$C = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2}$$

$$\theta = \sin^{-1} \frac{(A_y + B_y)}{C} = \sin^{-1} \left(\frac{C_y}{C} \right)$$

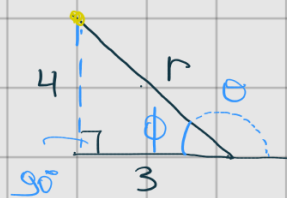
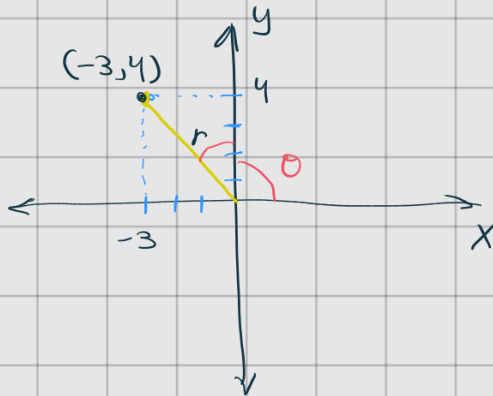
Example Problem

Let $x = 1$
 $y = \sqrt{3}$
 $r = ?$
 $\theta = ?$



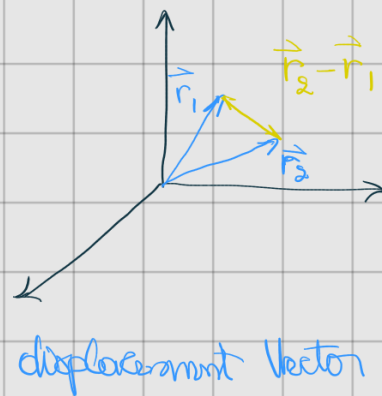
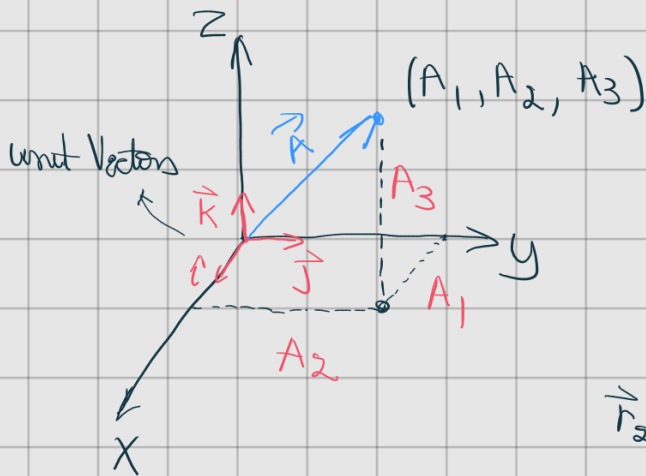
$r = \sqrt{1^2 + (\sqrt{3})^2} = 2$
 $\tan \theta = \sqrt{3}/1 = \sqrt{3}$
 $\theta = \tan^{-1}(\sqrt{3}) = 60^\circ$

$x = -3$
 $y = 4$
 $r = ?$
 $\theta = ?$



$r = \sqrt{3^2 + 4^2} = 5$
 $\phi + \theta = 180^\circ$
 $\cos \phi = 3/5$
 $\phi = \cos^{-1}(3/5) = 53^\circ$
 $\theta = 180^\circ - \phi = 180^\circ - 53^\circ = 127^\circ$

Cartesian Coordinates



$\vec{r}_2 - \vec{r}_1 = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}$

$\vec{A} = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$

$\vec{A} + \vec{B} = (A_1 + B_1)\vec{i} + (A_2 + B_2)\vec{j} + (A_3 + B_3)\vec{k}$

$k\vec{A} = kA_1\vec{i} + kA_2\vec{j} + kA_3\vec{k}$

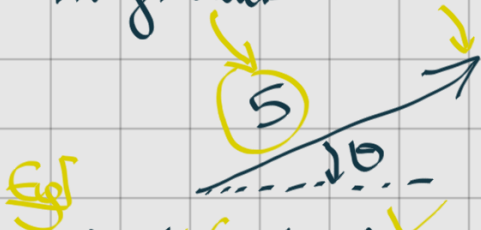
$|\vec{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2}$

Position Vectors

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Vector

magnitude + direction



Ex Velocity, displacement, gravity, force, Acceleration

$$\vec{v} = [v_x, v_y]$$

Magnitude

Temperature, mass, Volume, Speed

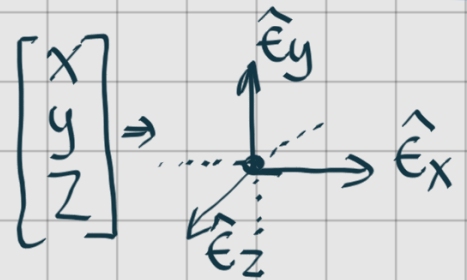
$$\sqrt{(A)^2 + (B)^2} = |\vec{v}|$$

Scalar multiplication

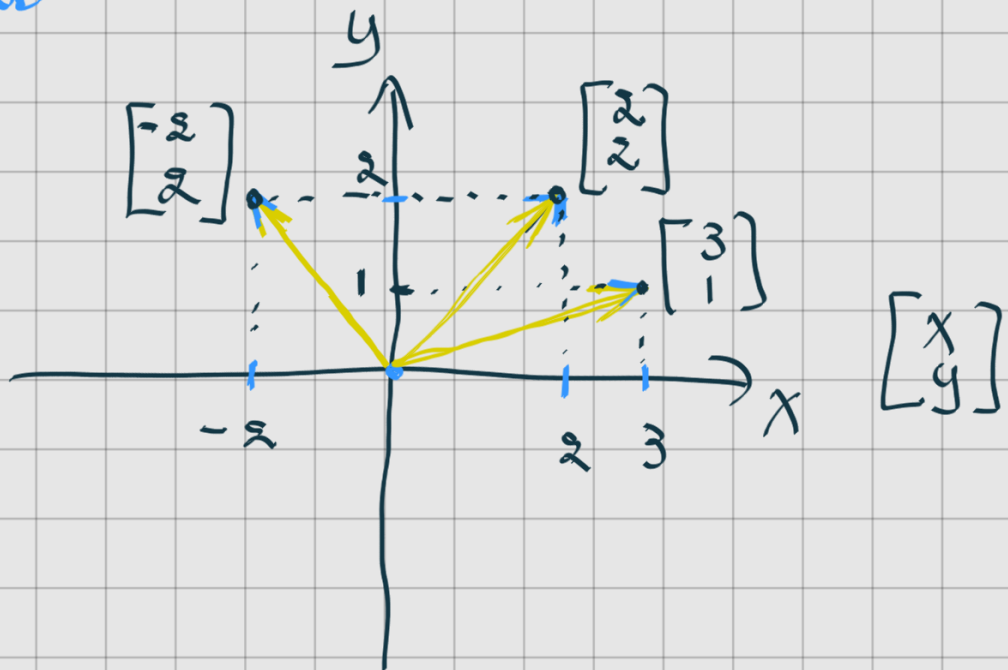
$$2 \vec{a} = 2 [2, 3] = [4, 3]$$

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2 + \dots}$$

magnitude



2D



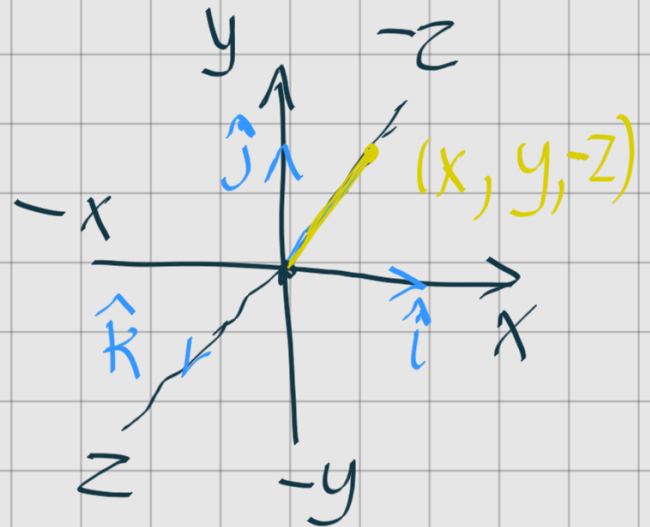
2D

$$\begin{bmatrix} x \\ y \end{bmatrix}$$



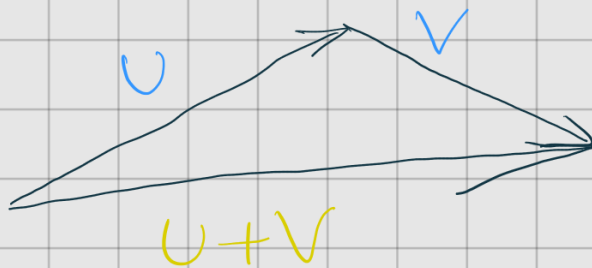
3D

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$



Vector Sum

complete triangles



Unit Vector Notation

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

2D

$$\vec{v} = v_x \hat{i} + v_y \hat{j} = |\vec{v}| \cos \theta \hat{i} + |\vec{v}| \sin \theta \hat{j}$$

$$|\vec{v}| = \sqrt{v_x^2 + v_y^2}$$



$$v_x = |\vec{v}| \cos \theta$$

$$v_y = |\vec{v}| \sin \theta$$

3D



$$\hat{i} = \langle 1, 0, 0 \rangle$$

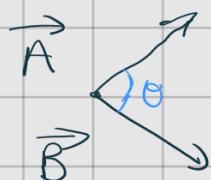
$$\hat{j} = \langle 0, 1, 0 \rangle$$

$$\hat{k} = \langle 0, 0, 1 \rangle$$

$$\vec{v} = v_x \hat{i} + v_y \hat{j} + v_z \hat{k}$$

$$\vec{v} = |\vec{v}| \cos \alpha \hat{i} + |\vec{v}| \cos \beta \hat{j} + |\vec{v}| \cos \gamma \hat{k}$$

Dot Product



$$\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta$$

$$= A_x B_x + A_y B_y + A_z B_z$$

$$\hat{i} \cdot \hat{i} = 1$$

$$\hat{i} \cdot \hat{j} = 0$$

$$\hat{j} \cdot \hat{j} = 1$$

$$\hat{j} \cdot \hat{k} = 0$$

$$\hat{k} \cdot \hat{k} = 1$$

$$\hat{i} \cdot \hat{k} = 0$$

Cross Product

$$\vec{A} \times \vec{B} = |\vec{A}| |\vec{B}| \sin \theta \cdot \hat{r}$$

magnitude direction

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{r_x \hat{i} + r_y \hat{j} + r_z \hat{k}}{\sqrt{r_x^2 + r_y^2 + r_z^2}}$$

$$\vec{C} = \vec{A} \times \vec{B} = |\vec{C}| \hat{r}$$

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$\vec{v} = |\vec{v}| \vec{u}$$

vector

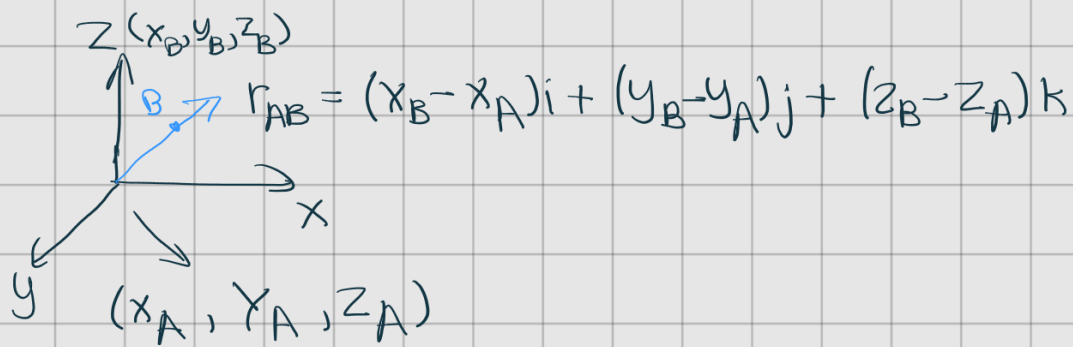
mag

direction

$$r_x = (x_B - x_A)$$

$$r_y = (y_B - y_A)$$

$$r_z = (z_B - z_A)$$



$$\hat{r} = \frac{\vec{r}_{AB}}{|\vec{r}|}$$

$$\vec{A} \times \vec{B} = |\vec{A}| |\vec{B}| \sin \theta, \quad \hat{r} = \vec{C}$$

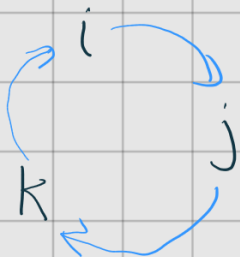
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\begin{aligned} \mathbf{i} \times \mathbf{j} &= \mathbf{k} \\ \mathbf{i} &= (1, 0, 0) \rightarrow A \\ \mathbf{j} &= (0, 1, 0) \rightarrow B \end{aligned}$$

$$\vec{C} = (0.0 - 0.1)\mathbf{i} + (0.0 - 1.0)\mathbf{j} + (1.1 - 0.0)\mathbf{k} = \mathbf{k}$$

$$\vec{C} = (A_y B_z - A_z B_y)\mathbf{i} + (A_z B_x - A_x B_z)\mathbf{j} + (A_x B_y - A_y B_x)\mathbf{k}$$

$$\vec{C} = \underset{\downarrow}{C_x} \mathbf{i} + \underset{\downarrow}{C_y} \mathbf{j} + \underset{\downarrow}{C_z} \mathbf{k}$$



$$\begin{array}{l|l} \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{j} \times \mathbf{i} = -\mathbf{k} \\ \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} \\ \mathbf{k} \times \mathbf{i} = \mathbf{j} & \mathbf{i} \times \mathbf{k} = -\mathbf{j} \end{array}$$

$$\mathbf{i} \times \mathbf{i} = 0$$

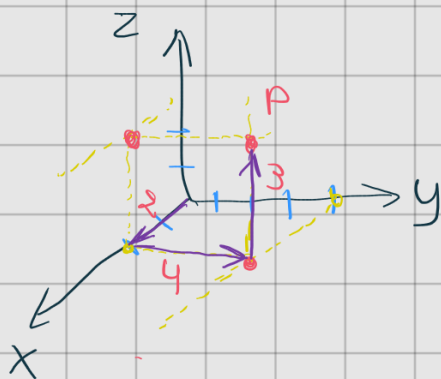
$$\mathbf{j} \times \mathbf{j} = 0$$

$$\mathbf{k} \times \mathbf{k} = 0$$

3D Coordinate Systems

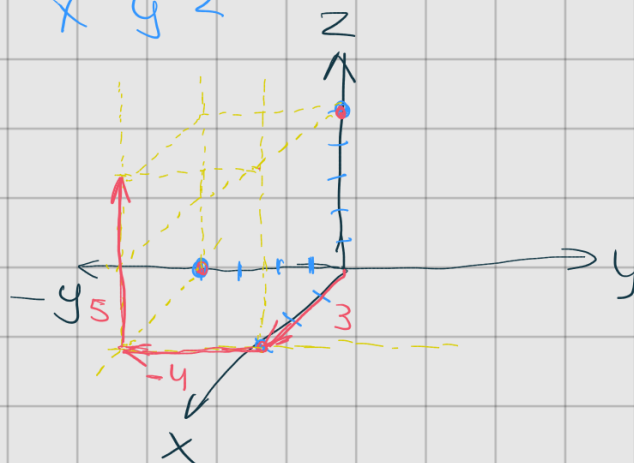
$$P(2, 4, 3)$$

$$P(x, y, z)$$



$$P(3, -4, 5)$$

$$x \ y \ z$$



Polar Coordinates

$$(3, 45^\circ)$$

$$R \ \theta$$

$$(2, \frac{3\pi}{4})$$

$$(2, 135^\circ)$$

$$\text{Polar} \\ (R, \theta)$$

$$(-2, 60^\circ)$$



Conversion

Polar $\rightarrow$ Rectangular Coordinates

$$(R, \theta) \rightarrow (x, y)$$

$$(4, 60^\circ) \rightarrow (2, 2\sqrt{3})$$

$$x = 4 \cos 60^\circ = 2$$

$$y = 4 \sin 60^\circ = 2\sqrt{3}$$

Rectangular $\rightarrow$ Polar

$$(x, y) \rightarrow (R, \theta)$$

$$(2, -4) \rightarrow (2, 63.4^\circ)$$

$$R = \sqrt{x^2 + y^2} = \sqrt{2^2 + (-4)^2} = 2$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = -\tan^{-1}\left(\frac{4}{2}\right)$$

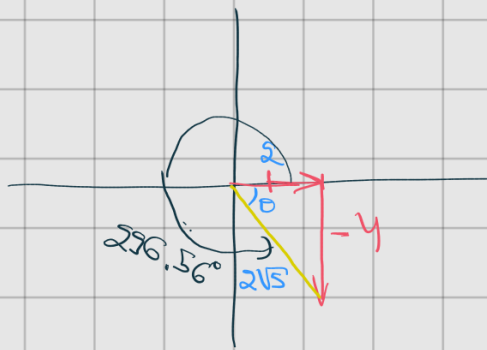
$$\theta = 63.4^\circ$$

$$R = 2\sqrt{5}$$

$$(2, -4)$$

$$\theta = 63.43^\circ$$

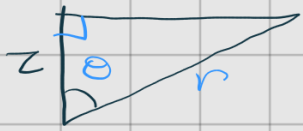
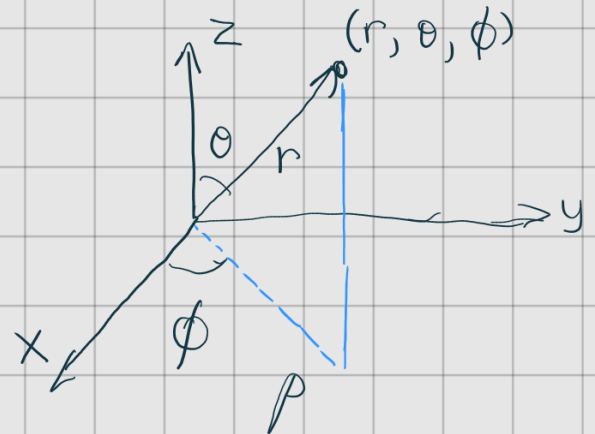
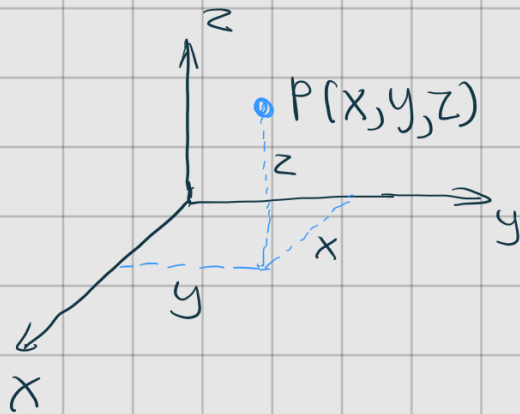
$$360^\circ - 63.43^\circ = 296.56^\circ$$



$$(2\sqrt{5}, 296.56^\circ)$$

Conversion between Cartesian and spherical Coordinate Systems

Electromagnetics



$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\cos \theta = \frac{z}{r} \rightarrow \theta = \cos^{-1} \frac{z}{r}$$

$$\tan \phi = \frac{y}{x} \rightarrow \phi = \tan^{-1} \frac{y}{x}$$

Cartesian Coordinate Vector

Spherical Coordinate Vector

$$\vec{A} = A_x \vec{a}_x + A_y \vec{a}_y + A_z \vec{a}_z$$

$$\vec{A} = A_r \vec{a}_r + A_\theta \vec{a}_\theta + A_\phi \vec{a}_\phi$$

$$P(r, \theta, \phi) \rightarrow P(x, y, z)$$

$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \cos^{-1} \left| \frac{z}{r} \right|$$

$$\phi = \tan^{-1} \left| \frac{y}{x} \right|$$

From Cartesian to Spherical

$$\begin{vmatrix} A_r \\ A_\theta \\ A_\phi \end{vmatrix} = \begin{vmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 1 \end{vmatrix} \begin{vmatrix} A_x \\ A_y \\ A_z \end{vmatrix}$$

From Spherical to Cartesian

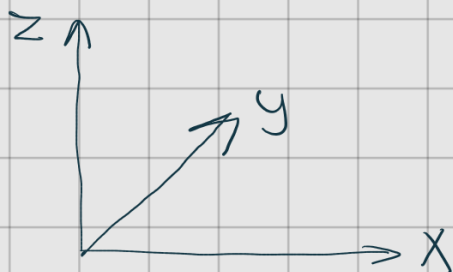
$$\begin{vmatrix} A_x \\ A_y \\ A_z \end{vmatrix} = \begin{vmatrix} \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\ \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\ \cos \theta & -\sin \theta & 1 \end{vmatrix} \begin{vmatrix} A_r \\ A_\theta \\ A_\phi \end{vmatrix}$$

$$\vec{F} = F_r \vec{a}_r + F_\theta \vec{a}_\theta + F_\phi \vec{a}_\phi \quad \left| \quad F_r = \frac{\cos \theta}{r^2} \quad F_\theta = \frac{\sin \theta}{r} \quad F_\phi = 0 \right.$$

Cartesian Coordinates



2D Axes
in plane



3D Axes
in Space

From Cartesian to Polar

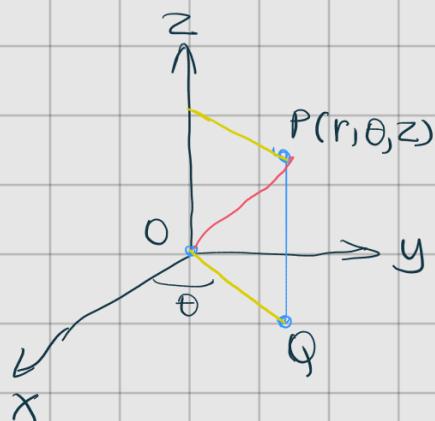
$$r = \sqrt{x^2 + y^2}$$

From Polar to Cartesian

$$x = r \cos(\theta)$$

$$y = r \sin(\theta)$$

Cylindrical Coordinates



Convert from Cylindrical
to Cartesian

$$x = r \cos(\theta), y = r \sin(\theta), z = z$$

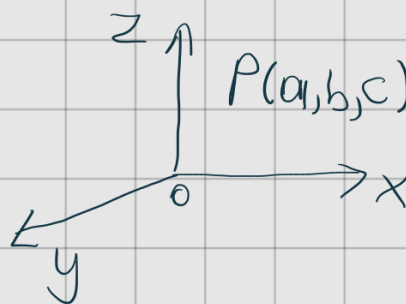
Convert from Cartesian to
Cylindrical

$$r = \sqrt{x^2 + y^2}, \theta = \left(\frac{y}{x} \right), z = z$$

Polar Coordinates



Cartesian Coordinates



Spherical Coordinate Systems



distances &
Angles

conversion from spherical to cartesian

$$x = \rho \sin(\phi) \cos(\theta)$$

$$y = \rho \sin(\phi) \sin(\theta)$$

$$z = \rho \cos(\phi)$$

convert from cartesian to spherical coordination

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \left(\frac{y}{x} \right)$$

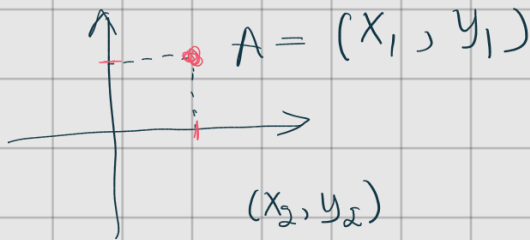
$$\varphi = \left(\frac{z}{\rho} \right)$$

System Coordinate

Slopes

$$m > 0$$

$$0^\circ < \theta < 90^\circ$$



$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = 0$$

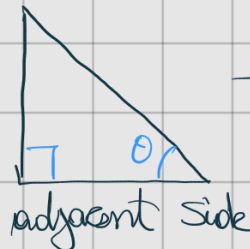
$$\theta = 0^\circ$$

$$m < 0$$

Tangent Function

$$90^\circ < \theta < 180^\circ$$

opposite Side



$$\tan \theta = \frac{\text{opposite Side}}{\text{adjacent Side}}$$

Point-slope Equation

$$y - y_1 = m(x - x_1)$$

General Equation of the Straight line $\theta = 90^\circ$

$$Ax + By + C = 0$$

Explicit Equation of the line

$$y = mx + b$$

Segmental Equation

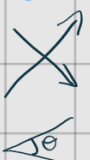
$$\frac{x}{a} + \frac{y}{b} = 1$$

Parametric Equation

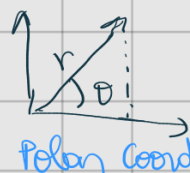
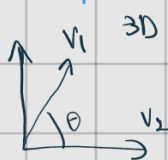
$$x = x_0 + at$$

$$y = y_0 + bt$$

dot Product



Cross Product



Polar Coord



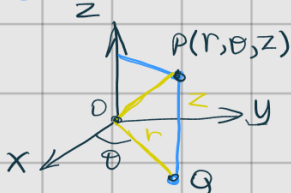
Cartesian Coord

$$r = \sqrt{x^2 + y^2}$$

Conversion

$$\theta = \left(\frac{y}{x} \right)$$

Cylindrical Coord



$$x = r \sin \theta$$

$$y = r \cos \theta$$

Conversion

$$z = z$$

Example

$$P_1(1, 2, -5)$$

$x_1 \ y_1 \ z_1$

$$P_2(4, 6, 7)$$

$x_2 \ y_2 \ z_2$

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$D = \sqrt{(4-1)^2 + (6-2)^2 + (7-(-5))^2} = 13$$

The coordinate System

- Cartesian / Rectangular
- Cylindrical or Circular
- Spherical

Dot Product

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

$$\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$$

$$y = \vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta = A_x B_x + A_y B_y + A_z B_z$$

Cross Product

$$\vec{A} \times \vec{B}$$

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

$$\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$$

$$= \hat{i}(A_y B_z - B_y A_z) - \hat{j}(A_x B_z - B_x A_z) + \hat{k}(A_x B_y - B_x A_y)$$

Scalar multiplication

$$\vec{y} = k \vec{A}$$

Basics of Vector

Representation of Vector

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

$$\vec{A} = (A_x, A_y, A_z)$$

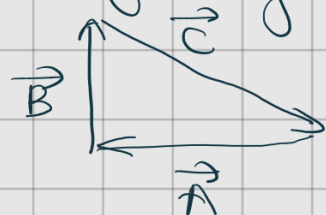
Magnitude of Vector

$$A = |\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Unit Vector

$$\hat{i}_A = \frac{\vec{A}}{|\vec{A}|}$$

Law of triangle



$$\vec{C} = \vec{A} + \vec{B}$$

$$\vec{y} = \vec{A} \times \vec{B} = |\vec{A}| |\vec{B}| \sin \theta$$

$$\vec{y} = \vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

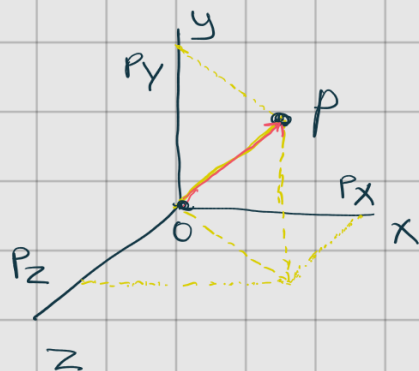
dot product

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$$

$$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$$

$$\hat{i} \cdot \hat{j} = 0, \hat{j} \cdot \hat{k} = 0, \hat{k} \cdot \hat{i} = 0$$

Cartesian Coordinate System



$$\vec{P} = P_x \hat{i} + P_y \hat{j} + P_z \hat{k}$$

Basis of Divergence

$$\text{Div}(\vec{F}) = \vec{\nabla} \cdot \vec{F} = \frac{\partial F}{\partial x} \hat{i} + \frac{\partial F}{\partial y} \hat{j} + \frac{\partial F}{\partial z} \hat{k}$$

$$\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$$

$$\text{Div}(\vec{F}) = \vec{\nabla} \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Basis of Curl

$$\text{Curl}(\vec{F}) = \vec{\nabla} \times \vec{F}$$

$$\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$$

$$\text{Curl}(\vec{F}) = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Function of Gauss Divergence Theorem

$$\oint P ds = \oint \vec{\nabla} \cdot \vec{P} dv$$

Cross product

$$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$$

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

$$\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}$$

$$\hat{j} \times \hat{i} = -\hat{k}, \hat{k} \times \hat{j} = -\hat{i}, \hat{i} \times \hat{k} = -\hat{j}$$

Projection of Vector

$$\vec{A} \cdot \vec{B} = AB \cos \theta \rightarrow A \cos \theta = \frac{\vec{A} \cdot \vec{B}}{B}$$

$$\vec{A} \cdot \vec{B} = AB \cos \theta \rightarrow B \cos \theta = \frac{\vec{A} \cdot \vec{B}}{A}$$

$$F_1 \cos \theta = \frac{\vec{F}_1 \cdot \vec{F}_2}{|\vec{F}_2|}$$

Basis of Gradient

$$\vec{\nabla} = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k}$$

$$\text{Grad}(F) = \vec{\nabla} F = \frac{\partial F}{\partial x} \hat{i} + \frac{\partial F}{\partial y} \hat{j} + \frac{\partial F}{\partial z} \hat{k}$$

$$\int \vec{\nabla} \cdot \vec{F} dv = \int \vec{F} \cdot d\vec{s}$$

$$\vec{\nabla} \times \vec{F} = \lim_{\Delta s \rightarrow 0} \frac{\int \vec{F} \cdot d\vec{l}}{\Delta s}$$

$$\int \vec{\nabla} \times \vec{F} \cdot d\vec{s} = \int \vec{F} \cdot d\vec{l}$$

Proof of Gauss Divergence Theorem

Divergence of Any function $\vec{P}$ can be calculated by

$$\text{Div}(\vec{P}) = \vec{\nabla} \cdot \vec{P} = \lim_{\Delta V \rightarrow 0} \frac{\oint \vec{P} \cdot d\vec{s}}{\Delta V}$$

$$\lim_{\Delta V \rightarrow 0} \vec{\nabla} \cdot \vec{P} \Delta V = \lim_{\Delta V \rightarrow 0} \oint \vec{P} \cdot d\vec{s}$$

total Volume

$$\oint \vec{P} \cdot d\vec{s} = \oint \vec{\nabla} \cdot \vec{P} dV$$

Function of Stokes theorem

$$\oint \vec{P} \cdot d\vec{l} = \int \vec{\nabla} \times \vec{P} \cdot d\vec{s}$$

Example

1 Specify the unit vector extending from the origin toward the point $G(\overset{x}{2}, \overset{y}{-2}, \overset{z}{-1})$

$$G(\overset{x}{2}, \overset{y}{-2}, \overset{z}{-1})$$

$$G = 2a_x - 2a_y - a_z$$

unit vector

$$a_G = \frac{G}{|G|} = \frac{a_x}{|G|} + \frac{a_y}{|G|} + \frac{a_z}{|G|} = \frac{2}{3}a_x - \frac{2}{3}a_y - \frac{1}{3}a_z$$

$$a_G = 0.667a_x - 0.667a_y - 0.333a_z$$

Magnitude

$$|G| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

$$|G| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = 3$$

2 Given points $M(\overset{x}{-1}, \overset{y}{2}, \overset{z}{1})$, $N(\overset{x}{3}, \overset{y}{-3}, \overset{z}{0})$, and $P(\overset{x}{-2}, \overset{y}{-3}, \overset{z}{-4})$,
Find: (a) R_{MN} ; (b) $R_{MN} + R_{MP}$; (c) $|r_M|$; (d) a_{MP} ; (e) $|2r_P - 3r_N|$.

$$(a) R_{MN} = r_N - r_M$$

$$r_M = -a_x + 2a_y + a_z$$

$$r_N = 3a_x - 3a_y + \cancel{0a_z} = 3a_x - 3a_y$$

$$R_{MN} = r_N - r_M = (3a_x - 3a_y) - (-a_x + 2a_y + a_z)$$

$$R_{MN} = 3a_x - 3a_y + a_x - 2a_y - a_z = a_x(3+1) - a_y(3+2) - a_z$$

$$R_{MN} = 4a_x - 5a_y - a_z$$

$$(b) R_{MN} + R_{MP}$$

$$R_{MN} = 4a_x - 5a_y - a_z$$

$$R_{MP} = r_P - r_M$$

$$r_M = -a_x + 2a_y + a_z$$

$$r_P = -2a_x - 3a_y - 4a_z$$

$$R_{MP} = r_P - r_M = (-2a_x - 3a_y - 4a_z) - (-a_x + 2a_y + a_z)$$

$$R_{MP} = -2a_x - 3a_y - 4a_z + a_x - 2a_y - a_z = a_x(-2+1) + a_y(-3-2) + a_z(-4-1)$$

$$R_{MP} = -a_x - 5a_y - 5a_z$$

$$R_{MN} + R_{MP} = (4a_x - 5a_y - a_z) + (-a_x - 5a_y - 5a_z)$$

$$R_{MN} + R_{MP} = 4a_x - 5a_y - a_z - a_x - 5a_y - 5a_z = a_x(4-1) + a_y(-5-5) + a_z(-1-5)$$

$$R_{MN} + R_{MP} = 4a_x - 5a_y - a_z + a_x - 5a_y - 5a_z = a_x(4-1) + a_y(-5-5) + a_z(-1-5)$$

$$R_{MN} + R_{MP} = 3a_x - 10a_y - 6a_z$$

(c) $|r_m|$

$$r_m = -a_x + 2a_y + a_z$$

magnitude

$$|r_m| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

$$|r_m| = \sqrt{(-1)^2 + (2)^2 + (1)^2} = \sqrt{1+4+1} = \sqrt{6}$$

$$|r_m| = \sqrt{6}$$

(d) a_{mp}

$$a_{mp} = \frac{R_{MP}}{|R_{MP}|} = \frac{r_p - r_m}{|R_{MP}|}$$

$$R_{MP} = r_p - r_m$$

$$r_m = -a_x + 2a_y + a_z$$

$$r_p = -2a_x - 3a_y - 4a_z$$

$$R_{MP} = r_p - r_m = (-2a_x - 3a_y - 4a_z) - (-a_x + 2a_y + a_z)$$

$$R_{MP} = r_p - r_m = -a_x - 5a_y - 5a_z$$

$$|R_{MP}| = \sqrt{a_x^2 + a_y^2 + a_z^2} = \sqrt{(-1)^2 + (-5)^2 + (-5)^2}$$

$$|R_{MP}| = \sqrt{1+25+25} = \sqrt{51} \approx 7.14$$

$$a_{mp} = \frac{R_{MP}}{|R_{MP}|} = \frac{-a_x - 5a_y - 5a_z}{7.14} = -\frac{1}{7.14}a_x - \frac{5}{7.14}a_y - \frac{5}{7.14}a_z$$

$$a_{mp} = -0.140 a_x - 0.700 a_y - 0.700 a_z$$

$$(E) |2r_p - 3r_N|$$

$$r_p = -2a_x - 3a_y - 4a_z$$

$$2r_p = 2[-2a_x - 3a_y - 4a_z]$$

$$2r_p = -4a_x - 6a_y - 8a_z$$

$$r_N = 3a_x - 3a_y$$

$$3r_N = 3[3a_x - 3a_y]$$

$$3r_N = 9a_x - 9a_y$$

$$2r_p - 3r_N = (-4a_x - 6a_y - 8a_z) - (9a_x - 9a_y)$$

$$2r_p - 3r_N = -4a_x - 6a_y - 8a_z - 9a_x + 9a_y = a_x(-4-9) + a_y(-6+9) - 8a_z$$

$$2r_p - 3r_N = -13a_x + 3a_y - 8a_z$$

The magnitude

$$|2r_p - 3r_N| = \sqrt{(-13)^2 + (3)^2 + (-8)^2} = \sqrt{242} = 15.56$$

(Co-ordinate Systems)
Cylindrical Coordinate Systems

Cylindrical to Cartesian Conversion (Vector)

$$A_x = [A_p \cos \phi - A_\phi \sin \phi]$$

$$A_y = [A_p \sin \phi + A_\phi \cos \phi]$$

$$A_z = A_z$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_p \\ A_\phi \\ A_z \end{bmatrix}$$

Conversion to cylindrical coordinates

$$z = z$$

$$x = \rho \cos \phi$$

$$y = \rho \sin \phi$$

$$\rho = \sqrt{x^2 + y^2}$$

$$\phi = \tan^{-1} \frac{y}{x}$$

$$z = z$$

Chapter 1 Calculus, Differentiation, and Integration

Source pages: Document 204159 pp. 1, 2, 6, 8, 10; Document 204511 pp. 18, 20, 22, 23

1.1 Power rule and negative powers

Rule.

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Examples rewritten cleanly:

$$\frac{d}{dx}(x^{-2}) = (-2)x^{-2-1} = -2x^{-3} = -\frac{2}{x^3}$$

$$\frac{d}{dx}(x^{-1}) = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$

For a reciprocal function:

$$\frac{d}{du}(u^{-1}) = (-1)u^{-2} = -\frac{1}{u^2}$$

Using the chain rule for $u = z^2 + 1$:

$$\frac{d}{dz}\left[\frac{1}{z^2 + 1}\right] = \frac{d}{dz}[(z^2 + 1)^{-1}] = -1(z^2 + 1)^{-2}(2z) = -\frac{2z}{(z^2 + 1)^2}$$

1.2 Integration by parts

Formula.

$$\int u \, dv = uv - \int v \, du$$

Useful selection order from the notes:

L I A T E

Logarithmic $\rightarrow$ Inverse trigonometric $\rightarrow$ Algebraic $\rightarrow$ Trigonometric $\rightarrow$ Exponential

Examples of choosing u and dv :

$$\int x e^x \, dx, \quad u = x, \quad dv = e^x \, dx, \quad v = e^x$$

$$\int \ln(x) \, x \, dx, \quad u = \ln(x), \quad dv = x \, dx$$

$$\int \sin^{-1}(x) \, dx, \quad u = \sin^{-1}(x), \quad dv = dx$$

$$\int x^2 \cos(x) \, dx, \quad u = x^2, \quad dv = \cos(x) \, dx$$

1.3 Integration by parts examples

Example 1: $\int \ln(x) \, dx$

$$\int \ln(x) \, dx = \int 1 \cdot \ln(x) \, dx$$

Choose

$$u = \ln(x), \quad dv = dx, \quad du = \frac{1}{x} \, dx, \quad v = x$$

Then

$$\int \ln(x) dx = x \ln(x) - \int x \left(\frac{1}{x}\right) dx = x \ln(x) - x + C$$

Example 2: $\int x \sin x dx$

$$\int x \sin x dx$$

Choose

$$u = x, \quad du = dx, \quad dv = \sin x dx, \quad v = -\cos x$$

Then

$$\int x \sin x dx = x(-\cos x) - \int (-\cos x)(1) dx = -x \cos x + \sin x + C$$

Example 3: $\int x^2 \ln x dx$

$$\int x^2 \ln x dx$$

Choose

$$u = \ln x, \quad du = \frac{1}{x} dx, \quad dv = x^2 dx, \quad v = \frac{x^3}{3}$$

Then

$$\int x^2 \ln x dx = \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \cdot \frac{1}{x} dx = \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2 dx$$

$$\boxed{\int x^2 \ln x dx = \frac{x^3}{3} \ln x - \frac{x^3}{9} + C}$$

Example 4: $\int e^x \sin x dx$

Let

$$I = \int e^x \sin x dx$$

Choose

$$u = \sin x, \quad dv = e^x dx, \quad du = \cos x dx, \quad v = e^x$$

Then

$$I = e^x \sin x - \int e^x \cos x dx$$

For the second integral, use integration by parts again:

$$\int e^x \cos x dx = e^x \cos x + \int e^x \sin x dx = e^x \cos x + I$$

Therefore

$$I = e^x \sin x - (e^x \cos x + I)$$

$$2I = e^x (\sin x - \cos x)$$

$$\boxed{\int e^x \sin x dx = \frac{e^x}{2} (\sin x - \cos x) + C}$$

1.4 Gaussian integrals and normalization

Standard forms:

$$\int_{-\infty}^{\infty} e^{-bx^2} dx = \sqrt{\frac{\pi}{b}}$$
$$\int_{-\infty}^{\infty} e^{-2ax^2} dx = \sqrt{\frac{\pi}{2a}}$$

If a normalized function has the form

$$\psi(x) = Ae^{-ax^2},$$

then

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$$
$$A^2 \int_{-\infty}^{\infty} e^{-2ax^2} dx = 1$$
$$A^2 \sqrt{\frac{\pi}{2a}} = 1$$

$$A = \left(\frac{2a}{\pi}\right)^{1/4}$$

1.5 Common trigonometric integral identities

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$\int \sin x \cos x dx = \int u du = \frac{u^2}{2} + C = \frac{\sin^2 x}{2} + C$$

1.6 Expectation value integral form

For a one-dimensional wavefunction on $0 \leq x \leq L$:

$$\langle x \rangle = \int_0^L \psi_n^*(x) x \psi_n(x) dx = \int_0^L x |\psi_n(x)|^2 dx$$

Normalization condition:

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$$

Chapter 2 Trigonometry and Identities

Source pages: Document 204159 pp. 2, 9; Document 204511 pp. 17-19

2.1 Right-triangle ratios

For a right triangle:

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \quad \tan \theta = \frac{\text{opposite}}{\text{adjacent}}$$

Reciprocal identities:

$$\csc \theta = \frac{1}{\sin \theta}, \quad \sec \theta = \frac{1}{\cos \theta}, \quad \cot \theta = \frac{1}{\tan \theta}$$

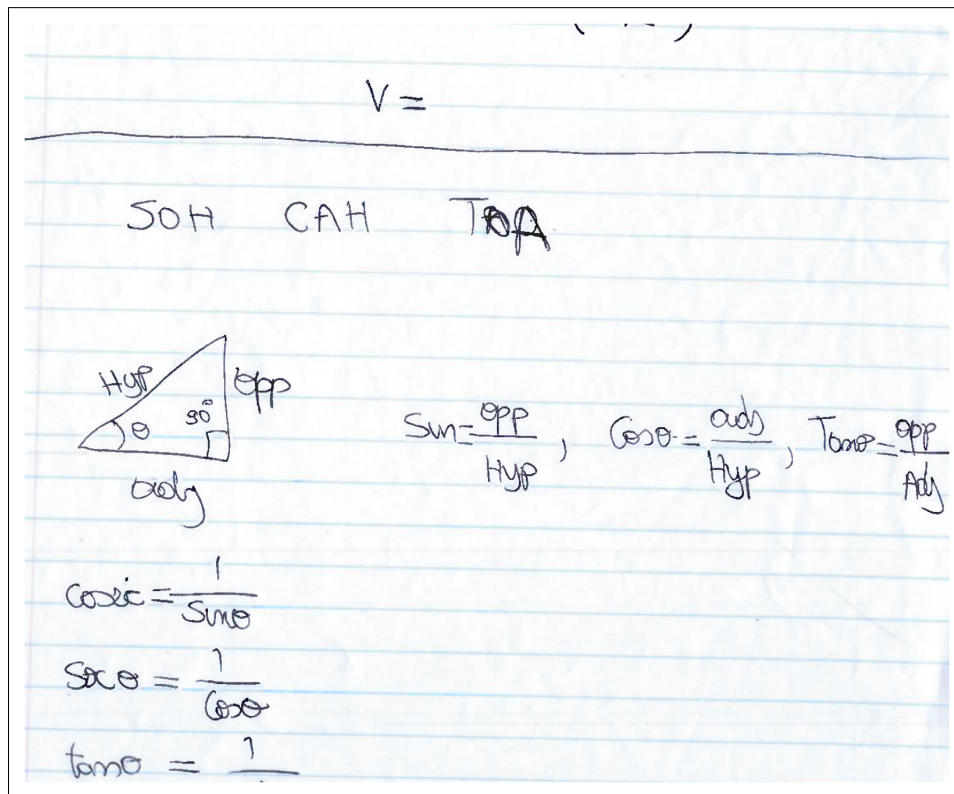


Figure 1: Original right-triangle SOH–CAH–TOA sketch reused from the handwritten notes.

2.2 Unit-circle sign pattern

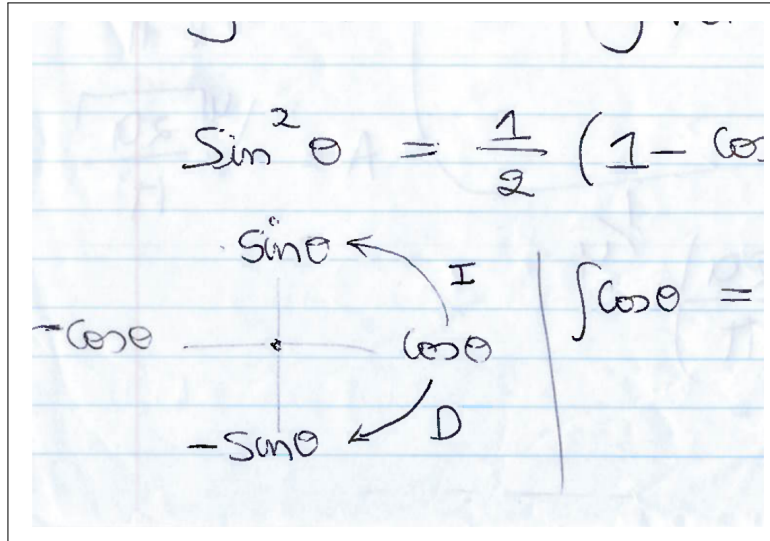


Figure 2: Original unit-circle sign sketch reused from the handwritten notes.

2.3 Even and odd identities

Odd functions:

$$\sin(-\theta) = -\sin \theta, \quad \tan(-\theta) = -\tan \theta, \quad \csc(-\theta) = -\csc \theta, \quad \cot(-\theta) = -\cot \theta$$

Even functions:

$$\cos(-\theta) = \cos \theta, \quad \sec(-\theta) = \sec \theta$$

2.4 Cofunction identities

$$30^\circ = \frac{\pi}{6} = \frac{180^\circ}{6}$$

$$90^\circ = \frac{\pi}{2} = \frac{180^\circ}{2}$$

$$\cos \theta = \sin(90^\circ - \theta)$$

$$\csc \theta = \sec(90^\circ - \theta)$$

$$\cot \theta = \tan(90^\circ - \theta)$$

$$\sin \theta = \cos(90^\circ - \theta)$$

2.5 Double-angle identities

$$\sin(2\theta) = 2 \sin \theta \cos \theta$$

$$\cos(2\theta) = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta$$

$$\tan(2\theta) = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

2.6 Half-angle identities

$$\begin{aligned}\sin\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1-\cos\theta}{2}} \\ \cos\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1+\cos\theta}{2}} \\ \tan\left(\frac{\theta}{2}\right) &= \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}\end{aligned}$$

2.7 Sum and difference identities

$$\begin{aligned}\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\ \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\ \tan(A + B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \\ \tan(A - B) &= \frac{\tan A - \tan B}{1 + \tan A \tan B}\end{aligned}$$

2.8 Product-to-sum identities

$$\begin{aligned}\sin A \sin B &= \frac{1}{2} [\cos(A - B) - \cos(A + B)] \\ \cos A \cos B &= \frac{1}{2} [\cos(A - B) + \cos(A + B)] \\ \sin A \cos B &= \frac{1}{2} [\sin(A + B) + \sin(A - B)]\end{aligned}$$

2.9 Useful derived identities

From

$$\cos(A + A) = \cos A \cos A - \sin A \sin A,$$

we get

$$\cos(2A) = \cos^2 A - \sin^2 A$$

Using

$$\cos^2 A + \sin^2 A = 1,$$

then

$$\begin{aligned}\cos^2 A &= 1 - \sin^2 A \\ \cos(2A) &= 1 - 2\sin^2 A \\ \sin^2 A &= \frac{1 - \cos(2A)}{2}\end{aligned}$$

How this pack is organized

Chapter	What is included
Calculus and trigonometry	Differentiation, integration by parts, Gaussian normalization, trigonometric identities used in later solutions.
Electrostatics	Coulomb law, electric field, charge density, divergence, Gauss law, coaxial cable/capacitor, annular disk, potential, dipole.
Magnetostatics	Biot-Savart law, straight wire, circular loop, Ampere law, toroid/current symmetry.
Waves and optics	Maxwell wave equation, critical angle, Brewster angle, double-slit spacing, phase/path difference, diffraction grating.
Atomic and quantum physics	Black-body radiation, photoelectric effect, Compton scattering, particle in a box, expectation value, hydrogen radial probability, angular momentum.

Chapter 1 Calculus and Trigonometry Tools

1.1 Differentiation rules used in Physics 5

Concept

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(x^{-2}) = -2x^{-3} = -\frac{2}{x^3}$$

$$\frac{d}{dx}(x^{-1}) = -x^{-2} = -\frac{1}{x^2}$$

For a composite reciprocal function:

$$\frac{d}{dz}[(z^2 + 1)^{-1}] = -(z^2 + 1)^{-2}(2z) = -\frac{2z}{(z^2 + 1)^2}$$

1.2 Integration by parts

Concept

$$\int u dv = uv - \int v du$$

Typical Physics 5 use:

$$\int x \cos(kx) dx, \quad \int x \sin(kx) dx, \quad \int x^2 \ln x dx.$$

Example: integral needed for expectation value

Evaluate

$$\int x \cos(kx) dx.$$

Choose

$$u = x, \quad dv = \cos(kx) dx.$$

Then

$$du = dx, \quad v = \frac{\sin(kx)}{k}.$$

Therefore

$$\int x \cos(kx) dx = \frac{x \sin(kx)}{k} - \int \frac{\sin(kx)}{k} dx = \frac{x \sin(kx)}{k} + \frac{\cos(kx)}{k^2} + C.$$

1.3 Gaussian integral and normalization

Concept

$$\int_{-\infty}^{\infty} e^{-bx^2} dx = \sqrt{\frac{\pi}{b}}$$

If

$$\psi(x) = Ae^{-ax^2},$$

then

$$1 = \int_{-\infty}^{\infty} |\psi(x)|^2 dx = A^2 \int_{-\infty}^{\infty} e^{-2ax^2} dx.$$

$$A^2 \sqrt{\frac{\pi}{2a}} = 1 \quad \Rightarrow \quad A = \left(\frac{2a}{\pi}\right)^{1/4}.$$

1.4 Trigonometric identities

Concept

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta), \quad \cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(2\theta) = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

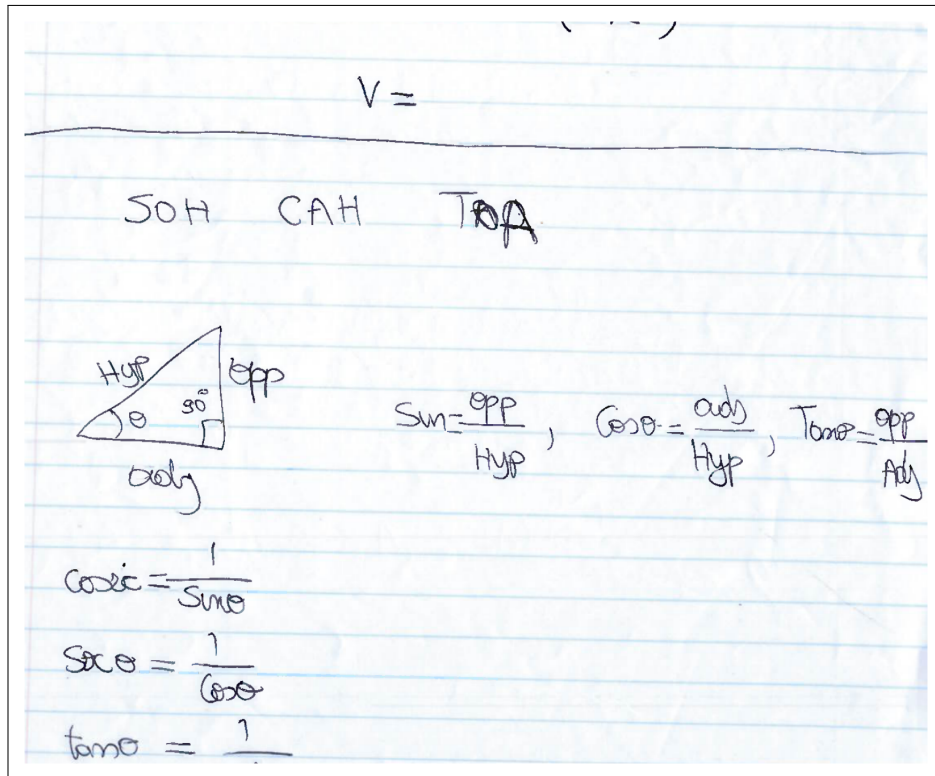


Figure 1: Original right-triangle SOH-CAH-TOA sketch reused from the concept notes.

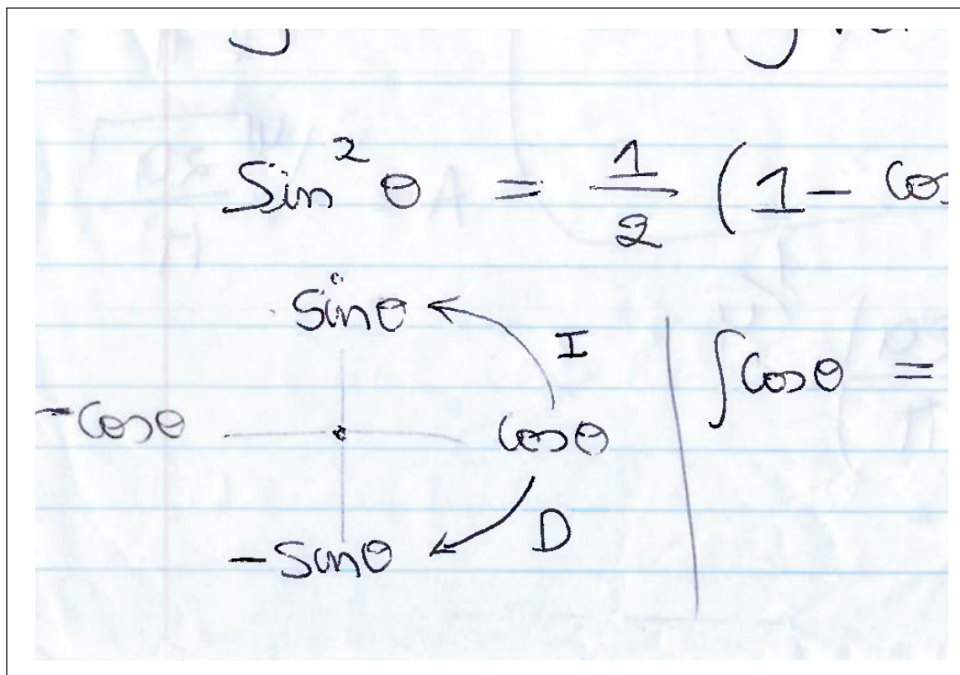


Figure 2: Original unit-circle sign sketch reused from the concept notes.

Chapter 2 Coulomb Law and Electric Field

1 Calculus and Trigonometry Tools

1.1 Differentiation rules used in Physics 5

Concept

$$\begin{aligned}\frac{d}{dx}(x^n) &= nx^{n-1}, \\ \frac{d}{dx}(x^{-2}) &= -2x^{-3} = -\frac{2}{x^3}, \\ \frac{d}{dx}(x^{-1}) &= -x^{-2} = -\frac{1}{x^2}.\end{aligned}$$

For a composite reciprocal function:

$$\begin{aligned}\frac{d}{dz} [(z^2 + 1)^{-1}] &= -(z^2 + 1)^{-2}(2z) \\ &= -\frac{2z}{(z^2 + 1)^2}.\end{aligned}$$

1.2 Integration by parts

Concept

$$\int u \, dv = uv - \int v \, du.$$

Common Physics 5 use:

$$\int x \cos(kx) \, dx, \quad \int x \sin(kx) \, dx, \quad \int x^2 \ln x \, dx.$$

Example: integral needed for expectation value

Evaluate

$$\int x \cos(kx) \, dx.$$

Choose

$$u = x, \quad dv = \cos(kx) \, dx.$$

Then

$$du = dx, \quad v = \frac{\sin(kx)}{k}.$$

Therefore

$$\begin{aligned}\int x \cos(kx) \, dx &= \frac{x \sin(kx)}{k} - \int \frac{\sin(kx)}{k} \, dx \\ &= \frac{x \sin(kx)}{k} + \frac{\cos(kx)}{k^2} + C.\end{aligned}$$

1.3 Gaussian integral and normalization

Concept

$$\int_{-\infty}^{\infty} e^{-bx^2} \, dx = \sqrt{\frac{\pi}{b}}.$$

If

$$\psi(x) = Ae^{-ax^2},$$

then

$$\begin{aligned}1 &= \int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = A^2 \int_{-\infty}^{\infty} e^{-2ax^2} \, dx, \\ A^2 \sqrt{\frac{\pi}{2a}} &= 1, \\ A &= \left(\frac{2a}{\pi}\right)^{1/4}.\end{aligned}$$

1.4 Trigonometric identities

Concept

$$\begin{aligned}\sin^2 \theta &= \frac{1}{2}(1 - \cos 2\theta), & \cos^2 \theta &= \frac{1}{2}(1 + \cos 2\theta), \\ \sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B, \\ \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B, \\ \tan(2\theta) &= \frac{2 \tan \theta}{1 - \tan^2 \theta}.\end{aligned}$$

Chapter 3 Electrostatics

Source pages: Document 204159 pp. 3, 7, 19, 23, 26; Document 204511 pp. 2, 3, 8, 9, 21, 24, 25, 27; Document 205011 pp. 1-9

3.1 Coulomb force in vector form

For charges q_1 and q_2 :

$$\mathbf{F} = k_e \frac{q_1 q_2}{R^3} \mathbf{R}$$

where

$$\mathbf{R} = \mathbf{r}_2 - \mathbf{r}_1, \quad R = |\mathbf{R}|.$$

If the same magnitude charge is used:

$$q_1 = q_2 = q_3 = q$$

then

$$\mathbf{F}_{12} = k_e \frac{q_1 q_2}{R_{12}^3} \mathbf{R}_{12} = k_e \frac{q^2}{R_{12}^3} \mathbf{R}_{12}$$

$$\mathbf{F}_{32} = k_e \frac{q_3 q_2}{R_{32}^3} \mathbf{R}_{32} = k_e \frac{q^2}{R_{32}^3} \mathbf{R}_{32}$$

$$\mathbf{F}_{\text{net}} = \mathbf{F}_{12} + \mathbf{F}_{32} = k_e q^2 \left[\frac{\mathbf{R}_{12}}{R_{12}^3} + \frac{\mathbf{R}_{32}}{R_{32}^3} \right]$$

Example component result from the notes:

$$\mathbf{F}_{\text{net}} = 5.43 \times 10^{-3} \hat{\mathbf{a}}_x - 3.18 \times 10^{-3} \hat{\mathbf{a}}_z \text{ N}$$

$$|\mathbf{F}_{\text{net}}| = \sqrt{F_{12}^2 + F_{32}^2} = 6.30 \times 10^{-3} \text{ N}$$

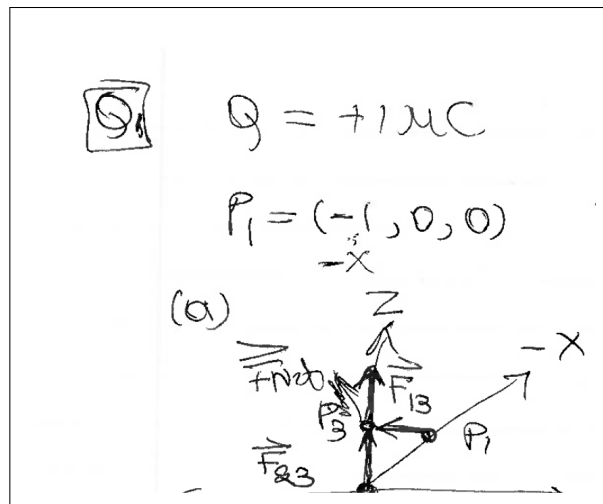


Figure 3: Original Coulomb-force vector sketch reused from the handwritten notes.

3.2 Electric field from several point charges

$$\mathbf{E}_{\text{net}} = k_e q \left[\frac{\mathbf{R}_1}{R_1^3} + \frac{\mathbf{R}_2}{R_2^3} + \frac{\mathbf{R}_3}{R_3^3} + \frac{\mathbf{R}_4}{R_4^3} \right]$$

With the point used in the notes:

$$\mathbf{R} = (1, 1, 1)$$

and source-charge positions

$$\mathbf{R}_1 = (1, 1, 0), \quad \mathbf{R}_2 = (-1, 1, 0), \quad \mathbf{R}_3 = (-1, -1, 0), \quad \mathbf{R}_4 = (1, -1, 0).$$

Relative position vectors:

$$\mathbf{R}_1 = \mathbf{R} - \mathbf{R}_1 = (1, 1, 1) - (1, 1, 0) = (0, 0, 1)$$

$$\mathbf{R}_2 = \mathbf{R} - \mathbf{R}_2 = (1, 1, 1) - (-1, 1, 0) = (2, 0, 1)$$

$$\mathbf{R}_3 = \mathbf{R} - \mathbf{R}_3 = (1, 1, 1) - (-1, -1, 0) = (2, 2, 1)$$

$$\mathbf{R}_4 = \mathbf{R} - \mathbf{R}_4 = (1, 1, 1) - (1, -1, 0) = (0, 2, 1)$$

Magnitudes:

$$|\mathbf{R}_1| = \sqrt{1^2} = 1$$

$$|\mathbf{R}_2| = \sqrt{2^2 + 1^2} = \sqrt{5}$$

$$|\mathbf{R}_3| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3$$

$$|\mathbf{R}_4| = \sqrt{2^2 + 1^2} = \sqrt{5}$$

Constant evaluation shown in the notes:

$$k_e q = 3 \mu\text{C} \times 9 \times 10^9 \text{ N m}^2/\text{C}^2 = 27 \text{ N m}^2/\text{C}$$

3.3 Potential of a point charge

Start from

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{a}}_r$$

$$d\mathbf{l} = dr \hat{\mathbf{a}}_r$$

$$\mathbf{E} \cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} dr$$

Potential:

$$V(r) = - \int_{\infty}^r \mathbf{E} \cdot d\mathbf{l}$$

$$V(r) = - \int_{\infty}^r \left(\frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \right) dr = - \frac{q}{4\pi\epsilon_0} \int_{\infty}^r r^{-2} dr$$

$$\int r^{-2} dr = -\frac{1}{r}$$

$$V(r) = -\frac{q}{4\pi\epsilon_0} \left[-\frac{1}{r} \right]_{\infty}^r = -\frac{q}{4\pi\epsilon_0} \left[-\frac{1}{r} - 0 \right]$$

$$\boxed{V(r) = \frac{q}{4\pi\epsilon_0 r}}$$

The work form is written the same way:

$$W = \int_{\infty}^r -F dx$$

$$W = -\frac{q}{4\pi\epsilon_0} \int_{\infty}^r x^{-2} dx = \frac{q}{4\pi\epsilon_0 r}$$

2.1 Point-charge vector method

Concept

For a source charge Q located at $\mathbf{r}'$ and an observation point $\mathbf{r}$,

$$\mathbf{R} = \mathbf{r} - \mathbf{r}', \quad R = |\mathbf{R}|.$$

Electric field:

$$\mathbf{E}(\mathbf{r}) = k_e \frac{Q}{R^3} \mathbf{R}.$$

For many point charges:

$$\mathbf{E}_{\text{net}} = \sum_i k_e \frac{Q_i}{R_i^3} \mathbf{R}_i.$$

Example: four identical point charges

Four identical charges $q = 3 \text{ nC}$ are at

$$(1, 1, 0), \quad (-1, 1, 0), \quad (-1, -1, 0), \quad (1, -1, 0).$$

Find $\mathbf{E}$ at

$$P(1, 1, 1).$$

Observation vector:

$$\mathbf{r} = (1, 1, 1).$$

Relative vectors:

$$\mathbf{R}_1 = (0, 0, 1), \quad \mathbf{R}_2 = (2, 0, 1),$$

$$\mathbf{R}_3 = (2, 2, 1), \quad \mathbf{R}_4 = (0, 2, 1).$$

Magnitudes:

$$R_1 = 1, \quad R_2 = \sqrt{5}, \quad R_3 = 3, \quad R_4 = \sqrt{5}.$$

Constant:

$$k_e q = (9 \times 10^9)(3 \times 10^{-9}) = 27.$$

Therefore

$$\mathbf{E} = 27 \left[\frac{(0, 0, 1)}{1^3} + \frac{(2, 0, 1)}{(\sqrt{5})^3} + \frac{(2, 2, 1)}{3^3} + \frac{(0, 2, 1)}{(\sqrt{5})^3} \right].$$

Final:

$$\boxed{\mathbf{E} = 6.83\hat{\mathbf{a}}_x + 6.83\hat{\mathbf{a}}_y + 32.8\hat{\mathbf{a}}_z \text{ V/m}}$$

2 Coulomb Law and Electric Field

2.1 Point-charge vector method

Concept

For a source charge Q_i located at $\mathbf{r}_i$ and an observation point $\mathbf{r}$:

$$\begin{aligned}\mathbf{R}_i &= \mathbf{r} - \mathbf{r}_i, \\ R_i &= |\mathbf{R}_i|, \\ \mathbf{E}_i &= k_e \frac{Q_i}{R_i^3} \mathbf{R}_i.\end{aligned}$$

For many point charges:

$$\mathbf{E}_{\text{net}} = \sum_i k_e \frac{Q_i}{R_i^3} \mathbf{R}_i.$$

Force between two charges:

$$\mathbf{F} = k_e \frac{q_1 q_2}{R^3} \mathbf{R}.$$

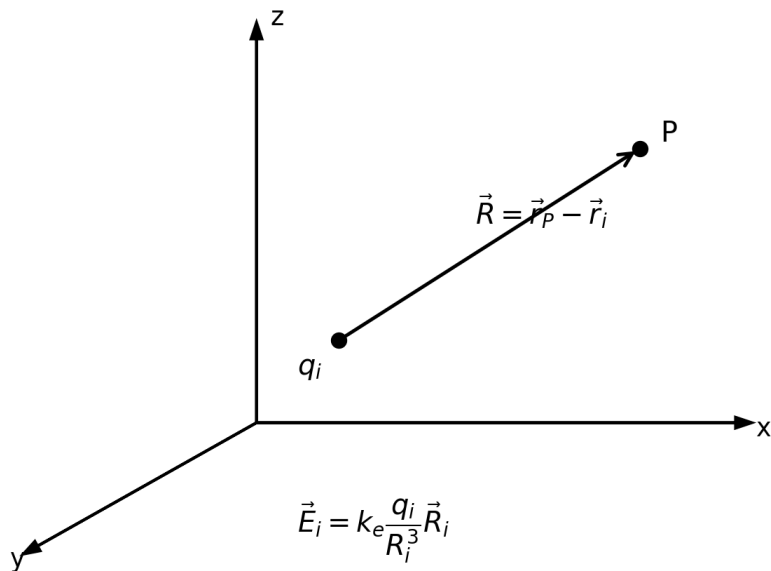


Figure 2.1: Clean machine diagram for the point-charge vector method.

Example: four identical point charges

Four identical charges $q = 3 \text{ nC}$ are at

$$(1, 1, 0), \quad (-1, 1, 0), \quad (-1, -1, 0), \quad (1, -1, 0).$$

Find $\mathbf{E}$ at

$$P(1, 1, 1).$$

Given

$$q = 3 \times 10^{-9} \text{ C}, \quad \mathbf{r} = (1, 1, 1), \quad k_e = 9 \times 10^9 \text{ N m}^2/\text{C}^2.$$

Formula

$$\mathbf{E}_{\text{net}} = k_e q \left[\frac{\mathbf{R}_1}{R_1^3} + \frac{\mathbf{R}_2}{R_2^3} + \frac{\mathbf{R}_3}{R_3^3} + \frac{\mathbf{R}_4}{R_4^3} \right].$$

Vectors

$$\begin{aligned} \mathbf{R}_1 &= (1, 1, 1) - (1, 1, 0) = (0, 0, 1), \\ \mathbf{R}_2 &= (1, 1, 1) - (-1, 1, 0) = (2, 0, 1), \\ \mathbf{R}_3 &= (1, 1, 1) - (-1, -1, 0) = (2, 2, 1), \\ \mathbf{R}_4 &= (1, 1, 1) - (1, -1, 0) = (0, 2, 1). \end{aligned}$$

Magnitudes

$$R_1 = 1, \quad R_2 = \sqrt{5}, \quad R_3 = 3, \quad R_4 = \sqrt{5}.$$

Substitution

$$\begin{aligned} k_e q &= (9 \times 10^9)(3 \times 10^{-9}) = 27, \\ \mathbf{E} &= 27 \left[\frac{(0, 0, 1)}{1^3} + \frac{(2, 0, 1)}{(\sqrt{5})^3} + \frac{(2, 2, 1)}{3^3} + \frac{(0, 2, 1)}{(\sqrt{5})^3} \right]. \end{aligned}$$

$$\mathbf{E} = 6.83\hat{\mathbf{a}}_x + 6.83\hat{\mathbf{a}}_y + 32.8\hat{\mathbf{a}}_z \text{ V/m}$$

2.2 Net force on a charge using superposition

Example: three-charge resultant force

Three equal positive charges q are fixed at

$$P_1(-1, 0, 0), \quad P_2(1, 0, 0), \quad P_3(0, 0, 1).$$

Find the resultant force on the charge at P_3 .

Formula

$$\mathbf{F}_{\text{net}} = k_e q^2 \left(\frac{\mathbf{R}_{13}}{R_{13}^3} + \frac{\mathbf{R}_{23}}{R_{23}^3} \right).$$

Vectors

$$\begin{aligned} \mathbf{R}_{13} &= \mathbf{r}_3 - \mathbf{r}_1 = (0, 0, 1) - (-1, 0, 0) = (1, 0, 1), \\ \mathbf{R}_{23} &= \mathbf{r}_3 - \mathbf{r}_2 = (0, 0, 1) - (1, 0, 0) = (-1, 0, 1), \\ R_{13} &= R_{23} = \sqrt{2}. \end{aligned}$$

Substitution

$$\begin{aligned}\mathbf{F}_{\text{net}} &= k_e q^2 \left[\frac{(1, 0, 1)}{(\sqrt{2})^3} + \frac{(-1, 0, 1)}{(\sqrt{2})^3} \right] \\ &= k_e q^2 \left[\frac{(0, 0, 2)}{(\sqrt{2})^3} \right] \\ &= \frac{k_e q^2}{\sqrt{2}} \hat{\mathbf{a}}_z.\end{aligned}$$

If $q = 1 \text{ C}$,

$$\mathbf{F}_{\text{net}} = \frac{9 \times 10^9}{\sqrt{2}} \hat{\mathbf{a}}_z.$$

$$\mathbf{F}_{\text{net}} = 6.36 \times 10^9 \hat{\mathbf{a}}_z \text{ N}$$

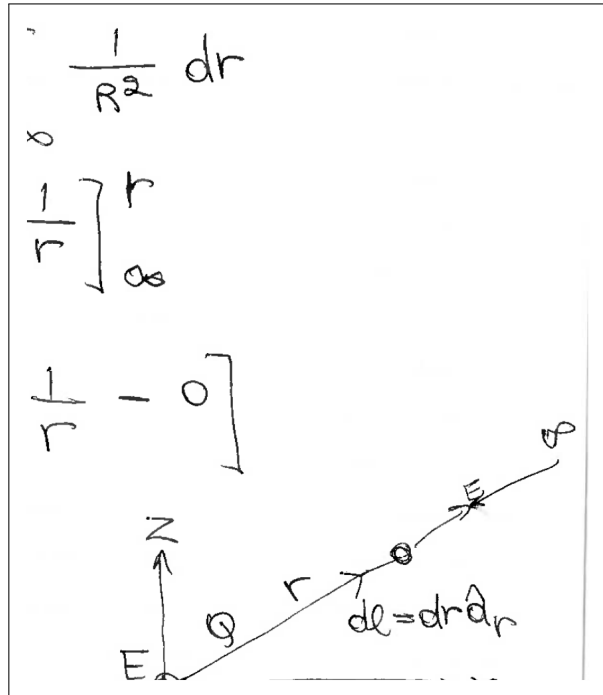


Figure 4: Original point-charge potential geometry reused from the handwritten notes.

3.4 Electric field from potential

General relation:

$$\mathbf{E} = -\nabla V$$

In cylindrical coordinates:

$$\nabla V = \frac{\partial V}{\partial \rho} \hat{\mathbf{a}}_\rho + \frac{1}{\rho} \frac{\partial V}{\partial \phi} \hat{\mathbf{a}}_\phi + \frac{\partial V}{\partial z} \hat{\mathbf{a}}_z$$

Therefore

$$\mathbf{E} = - \left[\frac{\partial V}{\partial \rho} \hat{\mathbf{a}}_\rho + \frac{1}{\rho} \frac{\partial V}{\partial \phi} \hat{\mathbf{a}}_\phi + \frac{\partial V}{\partial z} \hat{\mathbf{a}}_z \right]$$

Example from the notes:

$$\mathbf{E} = -10\hat{\mathbf{a}}_\rho + 17.32\hat{\mathbf{a}}_\phi + 24\hat{\mathbf{a}}_z \text{ V/m}$$

3.5 Divergence and enclosed charge

Use

$$\rho_v = \nabla \cdot \mathbf{D}$$

and

$$Q_{\text{enc}} = (\nabla \cdot \mathbf{D}) \Delta V.$$

Given

$$\mathbf{D} = e^{-x} \sin y \hat{\mathbf{a}}_x - e^{-x} \cos y \hat{\mathbf{a}}_y + 2z \hat{\mathbf{a}}_z,$$

then

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \frac{\partial}{\partial x} (e^{-x} \sin y) + \frac{\partial}{\partial y} (-e^{-x} \cos y) + \frac{\partial}{\partial z} (2z) \\ &= (-e^{-x} \sin y) + (e^{-x} \sin y) + 2 \end{aligned}$$

$$\nabla \cdot \mathbf{D} = 2$$

For

$$\Delta V = 10^{-3} \text{ m}^3,$$

$$Q_{\text{enc}} = 2(10^{-3}) = 2 \times 10^{-3} \text{ C}$$

3.6 Charged disk / annular sheet: electric field on the axis

Geometry used:

$$dq = \sigma dS, \quad dS = \rho d\rho d\phi, \quad dq = \sigma \rho d\rho d\phi$$
$$R = \sqrt{\rho^2 + z^2}, \quad \cos \theta = \frac{z}{R}$$

Differential field magnitude:

$$dE = k_e \frac{dq}{R^2}$$

Axis component:

$$dE_z = dE \cos \theta = k_e \frac{dq}{R^2} \left(\frac{z}{R} \right) = k_e \frac{z dq}{R^3}$$

Substitute:

$$dE_z = k_e \frac{\sigma z \rho d\rho d\phi}{(\rho^2 + z^2)^{3/2}}$$

Integrate from $\rho = a$ to $\rho = b$ and $\phi = 0$ to 2π :

$$E_z = \int dE_z = k_e \sigma z \int_0^{2\pi} \int_a^b \frac{\rho d\rho d\phi}{(\rho^2 + z^2)^{3/2}}$$
$$\int \frac{\rho d\rho}{(\rho^2 + z^2)^{3/2}} = -\frac{1}{\sqrt{\rho^2 + z^2}}$$
$$E_z = 2\pi k_e \sigma z \left[\frac{1}{\sqrt{a^2 + z^2}} - \frac{1}{\sqrt{b^2 + z^2}} \right]$$

Using

$$2\pi k_e = \frac{1}{2\epsilon_0},$$

$$\mathbf{E} = \frac{\sigma z}{2\epsilon_0} \left[\frac{1}{\sqrt{a^2 + z^2}} - \frac{1}{\sqrt{b^2 + z^2}} \right] \hat{\mathbf{a}}_z$$

Limit for an infinite sheet:

$$a \rightarrow 0, \quad b \rightarrow \infty$$
$$\mathbf{E} = \frac{\sigma z}{2\epsilon_0} \left[\frac{1}{\sqrt{0 + z^2}} - \frac{1}{\sqrt{\infty + z^2}} \right] \hat{\mathbf{a}}_z$$
$$\mathbf{E} = \frac{\sigma}{2\epsilon_0} \frac{z}{|z|} \hat{\mathbf{a}}_z$$

For the sheet direction:

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0} \hat{\mathbf{a}}_z \quad (z > 0), \quad \mathbf{E} = -\frac{\sigma}{2\epsilon_0} \hat{\mathbf{a}}_z \quad (z < 0)$$

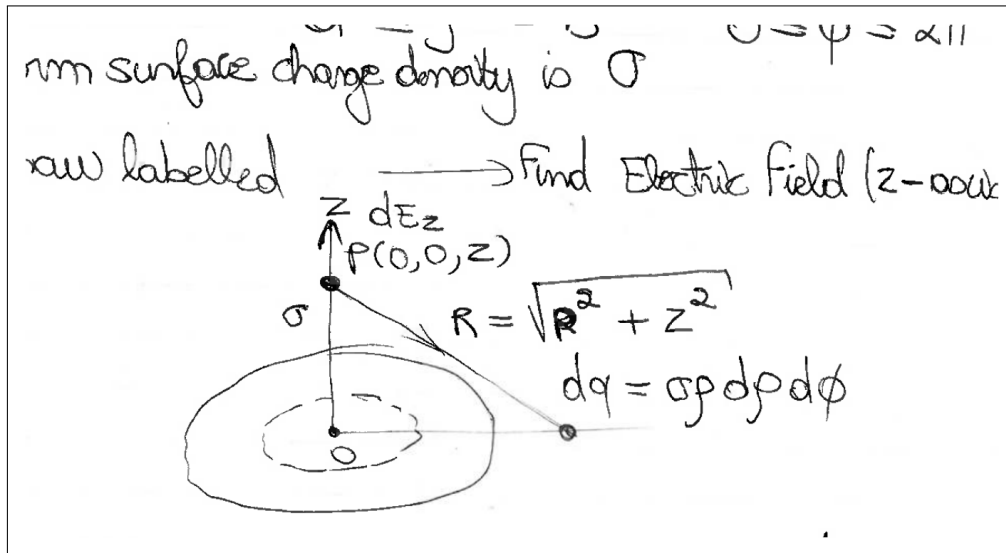


Figure 5: Original annular-sheet axis-field geometry reused from the handwritten notes.

3.7 Cylindrical volume charge: total charge

For cylindrical coordinates:

$$dV = \rho d\rho d\phi dz.$$

Total charge:

$$Q = \int_V \rho_v dV$$

Example charge density from the notes:

$$\rho_v = -2 \times 10^{-6} e^{-5\rho^2 z} \text{ C/m}^3$$

with limits

$$0 \leq \rho \leq 0.01 \text{ m}, \quad 0 \leq \phi \leq 2\pi, \quad 0.01 \leq z \leq 0.03 \text{ m}.$$

Thus

$$Q = \int_{0.01}^{0.03} \int_0^{2\pi} \int_0^{0.01} \left(-2 \times 10^{-6} e^{-5\rho^2 z} \right) \rho d\rho d\phi dz$$

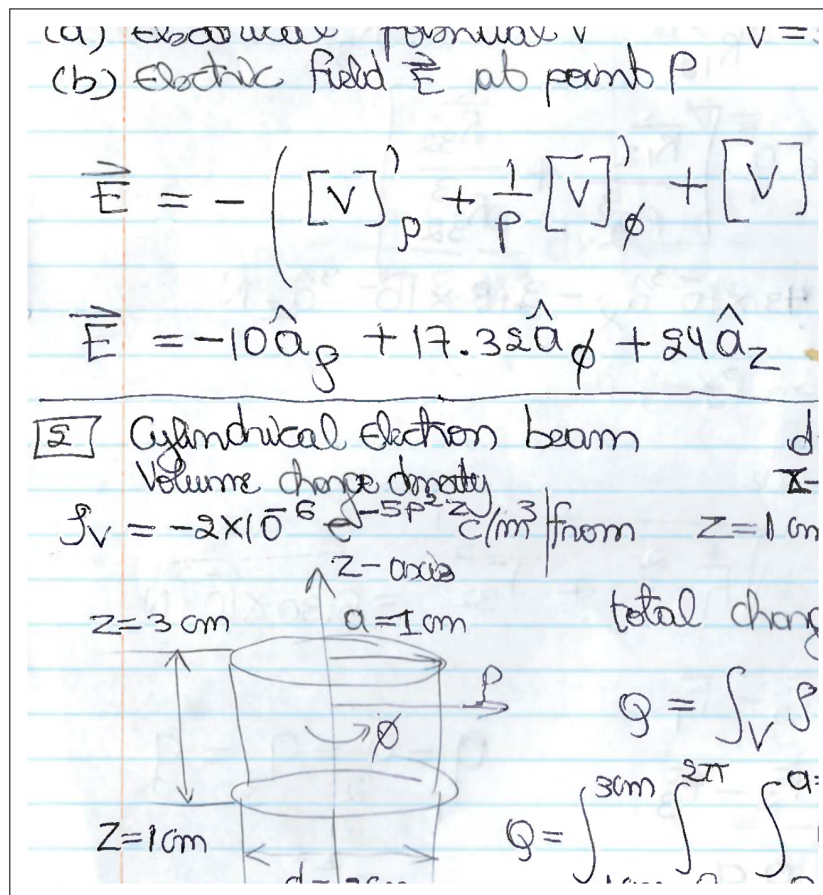


Figure 6: Original cylindrical electron-beam sketch reused from the handwritten notes.

3.8 Cylindrical charge density: Gauss-law setup

Given

$$\rho_v = a\rho^2, \quad 0 < \rho < b,$$

then

$$Q_{\text{enc}} = \int_V \rho_v dV = \int_0^L \int_0^{2\pi} \int_0^b (a\rho^2) \rho d\rho d\phi dz$$

$$Q_{\text{enc}} = \left[\int_0^L dz \right] \left[\int_0^{2\pi} d\phi \right] \left[\int_0^b a\rho^3 d\rho \right]$$

$$Q_{\text{enc}} = L(2\pi) \left(\frac{ab^4}{4} \right)$$

$$Q_{\text{enc}} = \frac{\pi Lab^4}{2} \text{ C}$$

For $0 < \rho < b$, Gauss's law gives

$$\frac{Q_{\text{enc}}}{\epsilon_0} = E(2\pi\rho L)$$

$$\mathbf{E} = \frac{a\rho^3}{4\epsilon_0} \hat{a}_{\rho}$$

3.9 Cylindrical capacitor: field, potential, capacitance, energy

Electric field between cylindrical conductors:

$$\mathbf{E} = \frac{\sigma a}{\varepsilon_0 \rho} \hat{\mathbf{a}}_\rho$$

Potential difference:

$$\begin{aligned} V_{AB} &= V_B - V_A \\ V_{AB} &= \int_A^B |\mathbf{E}| d\rho = \int_a^b \frac{\sigma a}{\varepsilon_0 \rho} d\rho \\ V_{AB} &= \frac{\sigma a}{\varepsilon_0} \int_a^b \frac{1}{\rho} d\rho = \frac{\sigma a}{\varepsilon_0} [\ln(\rho)]_a^b \end{aligned}$$

$$V_{AB} = \frac{\sigma a}{\varepsilon_0} \ln\left(\frac{b}{a}\right)$$

Total charge on the inner conductor:

$$A = 2\pi aL$$

$$Q_{\text{enc}} = \sigma A = \sigma(2\pi aL) = 2\pi\sigma aL$$

Capacitance:

$$\begin{aligned} C &= \frac{Q}{V} \\ C &= \frac{\sigma(2\pi aL)}{\frac{\sigma a}{\varepsilon_0} \ln\left(\frac{b}{a}\right)} \\ C &= \frac{2\pi\varepsilon_0 L}{\ln(b/a)} \end{aligned}$$

Energy by field integral:

$$\begin{aligned} W_E &= \frac{1}{2} \int_V \varepsilon_0 |\mathbf{E}|^2 dV \\ W_E &= \frac{1}{2} \int_0^L \int_0^{2\pi} \int_a^b \varepsilon_0 \left(\frac{\sigma a}{\varepsilon_0 \rho}\right)^2 \rho d\rho d\phi dz \\ W_E &= \frac{a^2 \sigma^2}{2\varepsilon_0} \left[\int_0^L dz \right] \left[\int_0^{2\pi} d\phi \right] \left[\int_a^b \frac{1}{\rho} d\rho \right] \\ W_E &= \frac{\pi a^2 \sigma^2 L}{\varepsilon_0} \ln\left(\frac{b}{a}\right) \end{aligned}$$

Energy using QV :

$$\begin{aligned} QV &= (2\pi\sigma aL) \left[\frac{\sigma a}{\varepsilon_0} \ln\left(\frac{b}{a}\right) \right] \\ QV &= \frac{2\pi a^2 L \sigma^2}{\varepsilon_0} \ln\left(\frac{b}{a}\right) \\ W &= \frac{1}{2} QV = \frac{\pi a^2 L \sigma^2}{\varepsilon_0} \ln\left(\frac{b}{a}\right) \end{aligned}$$

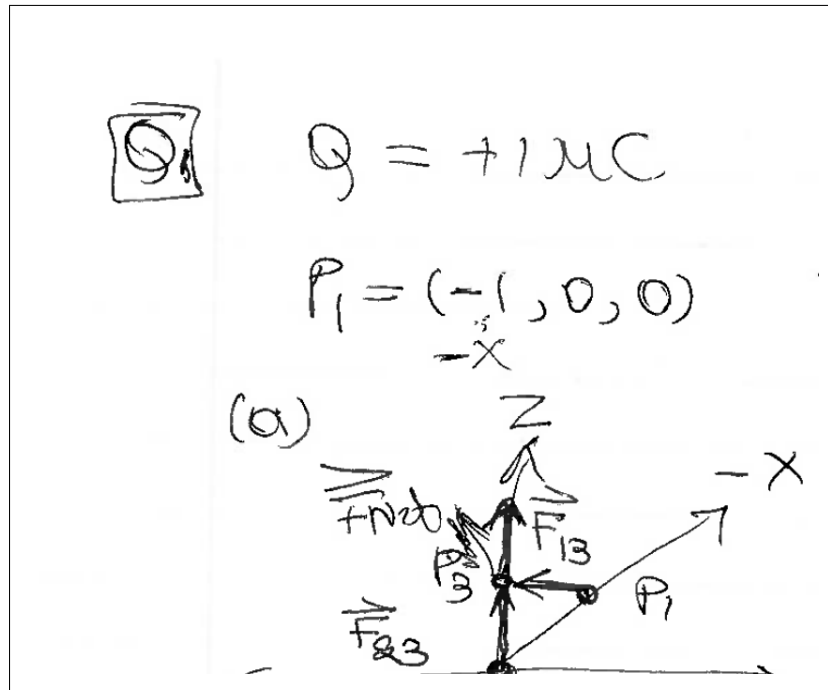


Figure 3: Original Coulomb/vector sketch reused from the concept notes.

Chapter 3 Charge Density, Divergence, and Total Charge

3.1 Total charge from volume charge density

Concept

In cylindrical coordinates:

$$dV = \rho \, d\rho \, d\phi \, dz.$$

Total charge:

$$Q = \iiint_V \rho_v \, dV.$$

For charge per unit length, integrate over the cross-section only:

$$\lambda = \int_0^{2\pi} \int_0^b \rho_v \rho \, d\rho \, d\phi.$$

Example: cylindrical electron beam charge per meter

Given

$$\rho_v = -\frac{0.1}{\rho^2 + 10^{-8}} \text{ pC/m}^3, \quad 0 < \rho < 3 \times 10^{-4} \text{ m}.$$

Charge per meter:

$$\lambda = \int_0^{2\pi} \int_0^{3 \times 10^{-4}} \left[-\frac{0.1}{\rho^2 + 10^{-8}} \right] \rho \, d\rho \, d\phi.$$

$$\lambda = 2\pi(-0.1) \int_0^{3 \times 10^{-4}} \frac{\rho}{\rho^2 + 10^{-8}} \, d\rho.$$

Use

$$\int \frac{\rho}{\rho^2 + c} \, d\rho = \frac{1}{2} \ln(\rho^2 + c).$$

Then

$$\lambda = -0.1\pi \left[\ln(\rho^2 + 10^{-8}) \right]_0^{3 \times 10^{-4}}.$$

$$\lambda = -0.1\pi \ln \left(\frac{(3 \times 10^{-4})^2 + 10^{-8}}{10^{-8}} \right) \text{ pC/m.}$$

$$\lambda = -0.722 \text{ pC/m}$$

If electron velocity is

$$v = 5 \times 10^7 \text{ m/s,}$$

beam current is

$$I = \lambda v = (-0.722 \times 10^{-12})(5 \times 10^7) = -3.61 \times 10^{-5} \text{ C/s.}$$

$$I = -3.61 \times 10^{-5} \text{ A}$$

3.2 Divergence method for enclosed charge

Concept

$$\rho_v = \nabla \cdot \mathbf{D}$$

Small-volume approximation:

$$Q_{\text{enc}} \approx (\nabla \cdot \mathbf{D})\Delta V.$$

Example: divergence at an incremental volume

Given

$$\mathbf{D} = e^{-x} \sin y \hat{\mathbf{a}}_x - e^{-x} \cos y \hat{\mathbf{a}}_y + 2z \hat{\mathbf{a}}_z.$$

Then

$$\nabla \cdot \mathbf{D} = \frac{\partial}{\partial x}(e^{-x} \sin y) + \frac{\partial}{\partial y}(-e^{-x} \cos y) + \frac{\partial}{\partial z}(2z).$$

$$\nabla \cdot \mathbf{D} = -e^{-x} \sin y + e^{-x} \sin y + 2 = 2.$$

For

$$\Delta V = 10^{-3} \text{ m}^3,$$

$$Q_{\text{enc}} = (2)(10^{-3}) = 2 \times 10^{-3} \text{ C.}$$

$$Q_{\text{enc}} = 2.00 \times 10^{-3} \text{ C}$$

Chapter 4 Gauss Law, Coaxial Conductors, and Capacitors

4.1 Gauss law pattern

Concept

Gauss law in terms of $\mathbf{D}$:

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_{\text{enc}}.$$

Since

$$\mathbf{D} = \varepsilon_0 \mathbf{E},$$

then

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q_{\text{enc}}}{\varepsilon_0}.$$

For cylindrical symmetry:

$$d\mathbf{S} = \rho d\phi dz \hat{\mathbf{a}}_\rho.$$

4.2 Coaxial cable electric field

Concept

Inner conductor radius a , outer conductor inner radius b , surface charge density σ on the inner conductor. For $a < \rho < b$:

$$Q_{\text{enc}} = \sigma(2\pi aL).$$

Gauss law:

$$D_\rho(2\pi\rho L) = 2\pi aL\sigma.$$

Therefore

$$\mathbf{D} = \frac{a\sigma}{\rho}\hat{\mathbf{a}}_\rho, \quad \boxed{\mathbf{E} = \frac{a\sigma}{\varepsilon_0\rho}\hat{\mathbf{a}}_\rho}.$$

Regions:

$$\boxed{\mathbf{E} = 0 \quad (\rho < a)}, \quad \boxed{\mathbf{E} = \frac{a\sigma}{\varepsilon_0\rho}\hat{\mathbf{a}}_\rho \quad (a < \rho < b)}, \quad \boxed{\mathbf{E} = 0 \quad (\rho > b)}.$$

Example: coaxial capacitor potential, charge, capacitance, energy

Given

$$\mathbf{E} = \frac{\sigma a}{\varepsilon_0\rho}\hat{\mathbf{a}}_\rho, \quad a < \rho < b.$$

Potential difference:

$$V_{AB} = \int_a^b |\mathbf{E}| d\rho = \int_a^b \frac{\sigma a}{\varepsilon_0\rho} d\rho.$$

$$\boxed{V_{AB} = \frac{\sigma a}{\varepsilon_0} \ln\left(\frac{b}{a}\right)}$$

Total charge:

$$Q = \sigma A = \sigma(2\pi aL).$$

$$\boxed{Q = 2\pi aL\sigma}$$

Capacitance:

$$C = \frac{Q}{V_{AB}} = \frac{2\pi aL\sigma}{(\sigma a/\varepsilon_0) \ln(b/a)}.$$

$$\boxed{C = \frac{2\pi\varepsilon_0 L}{\ln(b/a)}}$$

Energy by field method:

$$W_E = \frac{1}{2} \iiint_V \varepsilon_0 E^2 dV.$$

$$W_E = \frac{1}{2} \int_0^L \int_0^{2\pi} \int_a^b \varepsilon_0 \left(\frac{\sigma a}{\varepsilon_0\rho}\right)^2 \rho d\rho d\phi dz.$$

$$W_E = \frac{\sigma^2 a^2}{2\varepsilon_0} (L)(2\pi) \int_a^b \frac{1}{\rho} d\rho.$$

$$\boxed{W_E = \frac{\pi a^2 \sigma^2 L}{\varepsilon_0} \ln\left(\frac{b}{a}\right)}$$

Also

$$\boxed{W_E = \frac{1}{2} QV}.$$

3 Charge Density, Divergence, and Total Charge

3.1 Divergence method for enclosed charge

Concept

Use Gauss's law in differential form:

$$\rho_v = \nabla \cdot \mathbf{D}.$$

For a very small volume ΔV :

$$Q_{\text{enc}} \approx (\nabla \cdot \mathbf{D})\Delta V.$$

In Cartesian coordinates:

$$\nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}.$$

Example: incremental charge from flux density

Given

$$\mathbf{D} = e^{-x} \sin y \hat{\mathbf{a}}_x - e^{-x} \cos y \hat{\mathbf{a}}_y + 2z \hat{\mathbf{a}}_z.$$

Find the approximate charge enclosed in $\Delta V = 10^{-3} \text{ m}^3$ at the origin.

Formula

$$Q_{\text{enc}} = (\nabla \cdot \mathbf{D})\Delta V.$$

Divergence

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \frac{\partial}{\partial x}(e^{-x} \sin y) + \frac{\partial}{\partial y}(-e^{-x} \cos y) + \frac{\partial}{\partial z}(2z) \\ &= -e^{-x} \sin y + e^{-x} \sin y + 2 \\ &= 2. \end{aligned}$$

Substitution

$$Q_{\text{enc}} = (2)(10^{-3}) = 2 \times 10^{-3} \text{ C}.$$

$$Q_{\text{enc}} = 2.00 \times 10^{-3} \text{ C}$$

Example: electric flux density and tiny-volume charge

Given

$$\mathbf{D} = 3x^2 \hat{\mathbf{a}}_x - 2zy^2 \hat{\mathbf{a}}_y + 4xy \hat{\mathbf{a}}_z \text{ C/m}^2.$$

Find the general volume charge density, its value at $P(2, 1, 5)$, and the approximate charge in a tiny volume.

Formula

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}.$$

Part (a): general expression

$$D_x = 3x^2, \quad D_y = -2zy^2, \quad D_z = 4xy.$$

Therefore

$$\begin{aligned}\rho_v &= \frac{\partial}{\partial x}(3x^2) + \frac{\partial}{\partial y}(-2zy^2) + \frac{\partial}{\partial z}(4xy) \\ &= 6x - 4zy + 0.\end{aligned}$$

$$\rho_v = 6x - 4zy \text{ C/m}^3$$

Part (b): value at $P(2, 1, 5)$

$$x = 2, \quad y = 1, \quad z = 5.$$

$$\begin{aligned}\rho_v(2, 1, 5) &= 6(2) - 4(5)(1) \\ &= 12 - 20 \\ &= -8.\end{aligned}$$

$$\rho_v(2, 1, 5) = -8 \text{ C/m}^3$$

Part (c): approximate enclosed charge

$$Q_{\text{enc}} \approx \rho_v \Delta V.$$

At $P(2, 1, 5)$:

$$Q_{\text{enc}} \approx (-8)\Delta V.$$

The uploaded image cuts off the full numerical value of ΔV , so the result is kept in substitution-ready form.

$$Q_{\text{enc}} = -8\Delta V \text{ C}$$

If the missing value is

$$\Delta V = 5 \times 10^{-n} \text{ m}^3,$$

then

$$Q_{\text{enc}} = (-8)(5 \times 10^{-n}) = -40 \times 10^{-n} \text{ C}.$$

$$Q_{\text{enc}} = -4.0 \times 10^{1-n} \text{ C}$$

3.2 Total charge from volume charge density

Concept

In cylindrical coordinates:

$$dV = \rho d\rho d\phi dz.$$

Total charge:

$$Q = \iiint_V \rho_v dV.$$

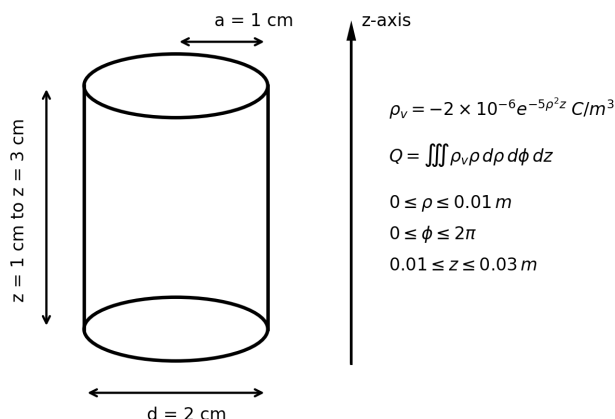


Figure 3.1: Clean cylindrical electron-beam geometry with radius, diameter, and z -limits.

Example: cylindrical electron beam

A cylindrical electron beam has diameter 2 cm and extends from $z = 1 \text{ cm}$ to $z = 3 \text{ cm}$. The charge density is

$$\rho_v = -2 \times 10^{-6} e^{-5\rho^2 z} \text{ C/m}^3.$$

Find the total charge.

Given

$$a = 1 \text{ cm} = 0.01 \text{ m}, \quad z_1 = 0.01 \text{ m}, \quad z_2 = 0.03 \text{ m}.$$

Formula

$$Q = \int_{z_1}^{z_2} \int_0^{2\pi} \int_0^a \rho_v \rho d\rho d\phi dz.$$

Substitution

$$Q = \int_{0.01}^{0.03} \int_0^{2\pi} \int_0^{0.01} \left(-2 \times 10^{-6} e^{-5\rho^2 z} \right) \rho d\rho d\phi dz.$$

Because $5\rho^2 z$ is very small over this region,

$$e^{-5\rho^2 z} \approx 1.$$

Thus

$$\begin{aligned} Q &\approx (-2 \times 10^{-6}) \int_{0.01}^{0.03} \int_0^{2\pi} \int_0^{0.01} \rho d\rho d\phi dz \\ &= (-2 \times 10^{-6}) \left[\frac{\rho^2}{2} \right]_0^{0.01} [\phi]_0^{2\pi} [z]_{0.01}^{0.03} \end{aligned}$$

$$= (-2 \times 10^{-6}) \left(\frac{(0.01)^2}{2} \right) (2\pi)(0.02).$$

$$Q \approx -1.26 \times 10^{-11} \text{ C}$$

4 Gauss Law, Coaxial Conductors, and Capacitors

4.1 Gauss law pattern

Concept

Gauss law in terms of $\mathbf{D}$:

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_{\text{enc}}.$$

Since

$$\mathbf{D} = \varepsilon_0 \mathbf{E},$$

then

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q_{\text{enc}}}{\varepsilon_0}.$$

For cylindrical symmetry:

$$d\mathbf{S} = \rho d\phi dz \hat{\mathbf{a}}_\rho.$$

4.2 Infinitely long cylindrical dielectric

Example: field outside a charged cylinder

An infinitely long cylindrical dielectric of radius b lies along the z -axis. The volume charge density is

$$\rho_v = a\rho^2.$$

Find the total charge in length L and the electric field outside the dielectric.

Part (a): total charge

$$\begin{aligned} Q_{\text{enc}} &= \int_0^L \int_0^{2\pi} \int_0^b (a\rho^2) \rho d\rho d\phi dz \\ &= \left[\int_0^L dz \right] \left[\int_0^{2\pi} d\phi \right] \left[\int_0^b a\rho^3 d\rho \right] \\ &= L(2\pi) \left[\frac{a\rho^4}{4} \right]_0^b \\ &= \frac{\pi a L b^4}{2}. \end{aligned}$$

$$Q_{\text{enc}} = \frac{\pi a L b^4}{2}$$

Part (b): electric field for $\rho > b$

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q_{\text{enc}}}{\epsilon_0}.$$

For cylindrical symmetry:

$$E_\rho(2\pi\rho L) = \frac{1}{\epsilon_0} \left(\frac{\pi a L b^4}{2} \right).$$

Therefore

$$E_\rho = \frac{ab^4}{4\epsilon_0\rho}.$$

$$\mathbf{E} = \frac{ab^4}{4\epsilon_0\rho} \hat{\mathbf{a}}_\rho \quad (\rho > b)$$

4.3 Coaxial capacitor method

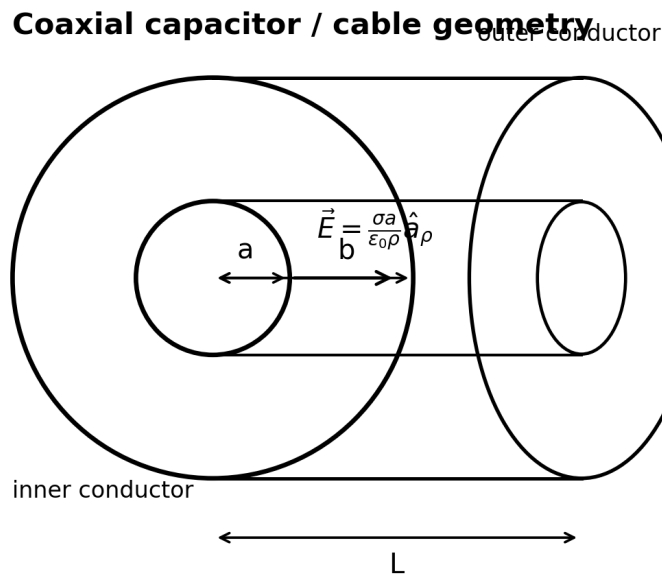


Figure 4.1: Clean coaxial capacitor geometry used for Gauss law and energy storage.

Concept

For a coaxial capacitor with inner radius a , outer radius b , length L , and surface charge density σ on the inner conductor:

$$Q = \sigma(2\pi aL).$$

For $a < \rho < b$:

$$\mathbf{E} = \frac{\sigma a}{\epsilon_0\rho} \hat{\mathbf{a}}_\rho.$$

Example: coaxial capacitor energy and voltage

Given

$$\mathbf{E} = \frac{\sigma a}{\varepsilon_0 \rho} \hat{\mathbf{a}}_\rho, \quad a < \rho < b.$$

Potential difference

$$\begin{aligned} V_{AB} &= \int_a^b |\mathbf{E}| d\rho = \int_a^b \frac{\sigma a}{\varepsilon_0 \rho} d\rho \\ &= \frac{\sigma a}{\varepsilon_0} [\ln \rho]_a^b. \end{aligned}$$

$$V_{AB} = \frac{\sigma a}{\varepsilon_0} \ln \left(\frac{b}{a} \right)$$

Total charge

$$Q = \sigma A = \sigma(2\pi a L).$$

$$Q = 2\pi a L \sigma$$

Capacitance

$$C = \frac{Q}{V_{AB}} = \frac{2\pi a L \sigma}{(\sigma a / \varepsilon_0) \ln(b/a)}.$$

$$C = \frac{2\pi \varepsilon_0 L}{\ln(b/a)}$$

Energy by field method

$$\begin{aligned} W_E &= \frac{1}{2} \iiint_V \varepsilon_0 E^2 dV \\ &= \frac{1}{2} \int_0^L \int_0^{2\pi} \int_a^b \varepsilon_0 \left(\frac{\sigma a}{\varepsilon_0 \rho} \right)^2 \rho d\rho d\phi dz \\ &= \frac{a^2 \sigma^2}{2\varepsilon_0} (L)(2\pi) \int_a^b \frac{1}{\rho} d\rho. \end{aligned}$$

$$W_E = \frac{\pi a^2 \sigma^2 L}{\varepsilon_0} \ln \left(\frac{b}{a} \right)$$

Also:

$$W_E = \frac{1}{2} QV.$$

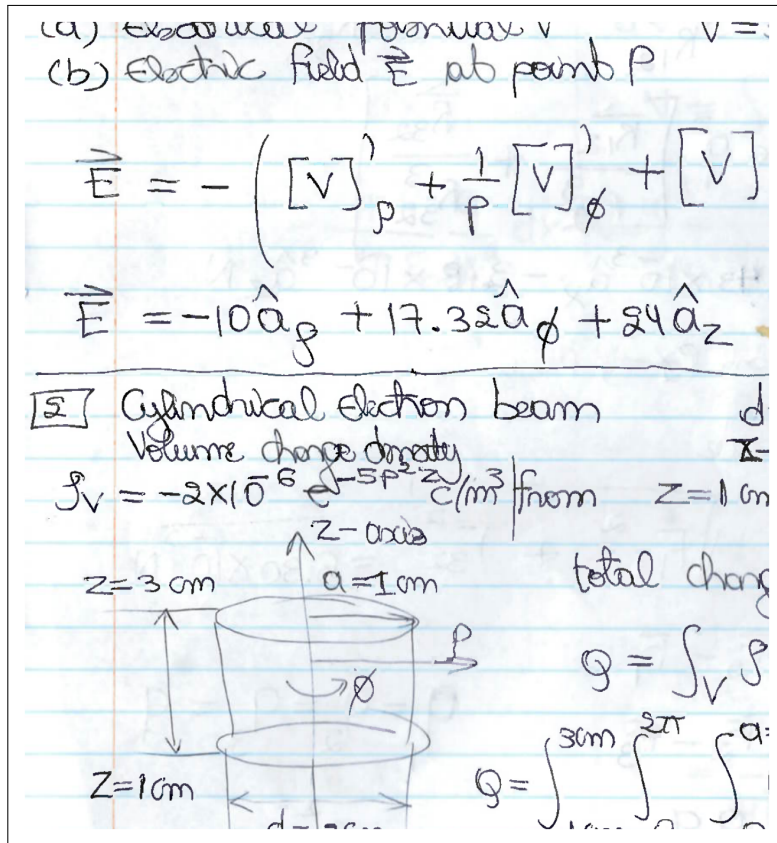


Figure 4: Original cylindrical beam/coaxial-style sketch reused from the concept notes.

Chapter 5 Annular Sheet and Plane Charge

5.1 Electric field on the axis of an annular charged sheet

Concept

For a surface charge density σ on

$$z = 0, \quad a \leq \rho \leq b, \quad 0 \leq \phi \leq 2\pi,$$

use

$$dS = \rho d\rho d\phi, \quad dq = \sigma \rho d\rho d\phi.$$

For point $P(0,0,z)$:

$$R = \sqrt{\rho^2 + z^2}, \quad \cos \theta = \frac{z}{R}.$$

Example: derive E_z for the annular sheet

$$dE = k_e \frac{dq}{R^2}.$$

Only the z components survive by symmetry:

$$dE_z = dE \cos \theta = k_e \frac{dq}{R^2} \frac{z}{R} = k_e \frac{z dq}{R^3}.$$

Substitute $dq = \sigma \rho d\rho d\phi$:

$$dE_z = k_e \frac{\sigma z \rho d\rho d\phi}{(\rho^2 + z^2)^{3/2}}.$$

Integrate:

$$E_z = k_e \sigma z \int_0^{2\pi} \int_a^b \frac{\rho d\rho d\phi}{(\rho^2 + z^2)^{3/2}}.$$

$$\int \frac{\rho d\rho}{(\rho^2 + z^2)^{3/2}} = -\frac{1}{\sqrt{\rho^2 + z^2}}.$$

Thus

$$E_z = 2\pi k_e \sigma z \left[\frac{1}{\sqrt{a^2 + z^2}} - \frac{1}{\sqrt{b^2 + z^2}} \right].$$

Since

$$2\pi k_e = \frac{1}{2\epsilon_0},$$

$$\mathbf{E} = \frac{\sigma z}{2\epsilon_0} \left[\frac{1}{\sqrt{a^2 + z^2}} - \frac{1}{\sqrt{b^2 + z^2}} \right] \hat{\mathbf{a}}_z$$

Plane limit:

$$a \rightarrow 0, \quad b \rightarrow \infty.$$

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0} \hat{\mathbf{a}}_z \quad (z > 0)$$

$$\mathbf{E} = -\frac{\sigma}{2\epsilon_0} \hat{\mathbf{a}}_z \quad (z < 0)$$

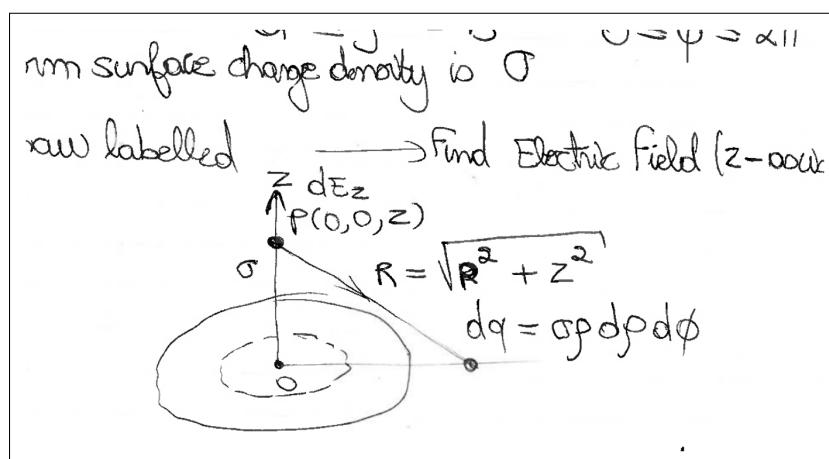


Figure 5: Original annular sheet/disk sketch reused from the concept notes.

Chapter 6 Electric Potential and Dipole

6.1 Point-charge potential

Concept

For a point charge Q at the origin:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{\mathbf{a}}_r.$$

Potential relative to infinity:

$$V(r) = -\int_{\infty}^r \mathbf{E} \cdot d\mathbf{l}.$$

Since

$$d\mathbf{l} = dr \hat{\mathbf{a}}_r,$$

$$V(r) = - \int_{\infty}^r \frac{Q}{4\pi\epsilon_0 r^2} dr.$$

$$V(r) = \frac{Q}{4\pi\epsilon_0 r}$$

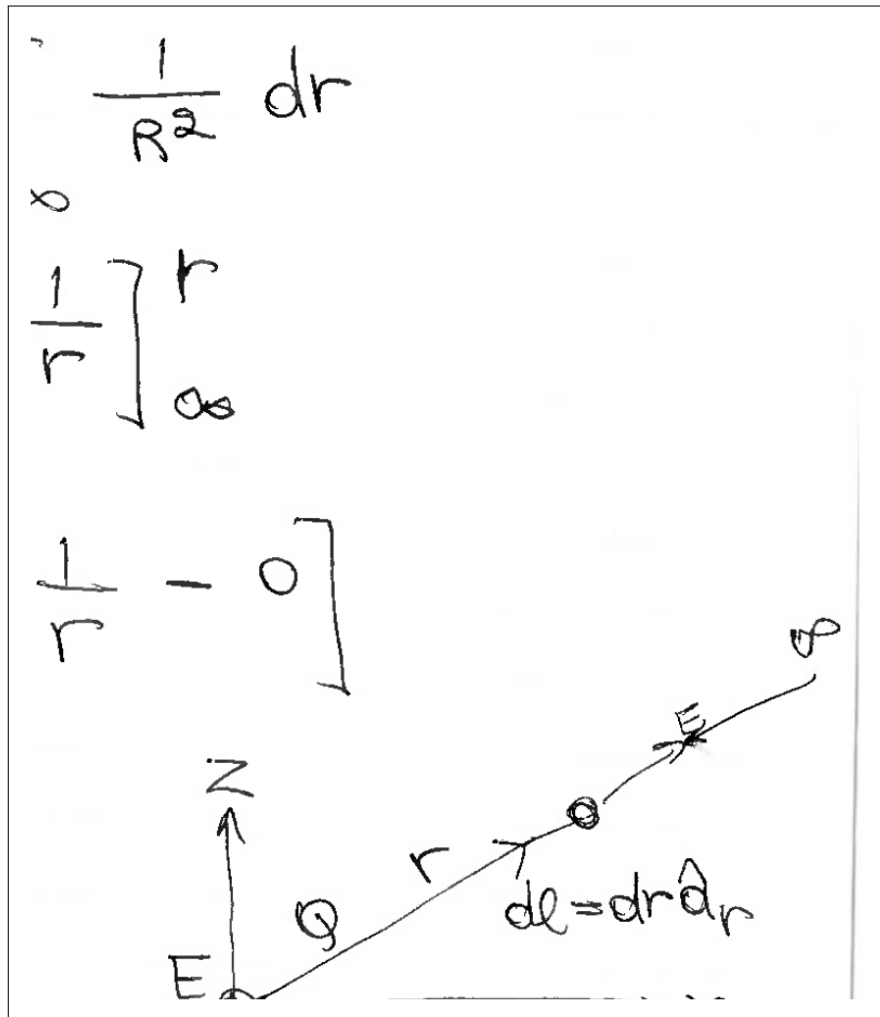


Figure 6: Original point-charge potential sketch reused from the concept notes.

6.2 Far-field electric dipole

Example: derive far-field potential and field of a dipole

Two charges:

$$+Q \text{ at } z = \frac{d}{2}, \quad -Q \text{ at } z = -\frac{d}{2}.$$

Potential at far point $P(r, \theta, \phi)$:

$$V = k_e Q \left(\frac{1}{R_1} - \frac{1}{R_2} \right) = k_e Q \left(\frac{R_2 - R_1}{R_1 R_2} \right).$$

For $r \gg d$:

$$R_2 - R_1 \approx d \cos \theta, \quad R_1 R_2 \approx r^2.$$

Therefore

$$V = \frac{k_e Q d \cos \theta}{r^2}$$

Let

$$p = Qd.$$

Then

$$V = \frac{k_e p \cos \theta}{r^2}.$$

Use spherical gradient:

$$\mathbf{E} = -\nabla V = -\left(\frac{\partial V}{\partial r} \hat{\mathbf{a}}_r + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\mathbf{a}}_\theta\right).$$
$$\frac{\partial V}{\partial r} = -\frac{2k_e p \cos \theta}{r^3}, \quad \frac{\partial V}{\partial \theta} = -\frac{k_e p \sin \theta}{r^2}.$$

Thus

$$\mathbf{E} = \frac{k_e p}{r^3} (2 \cos \theta \hat{\mathbf{a}}_r + \sin \theta \hat{\mathbf{a}}_\theta)$$

Chapter 7 Magnetostatics: Biot-Savart Law and Ampere Law

7.1 Biot-Savart law

Concept

Magnetic field intensity form:

$$d\mathbf{H} = \frac{I d\mathbf{l} \times \hat{\mathbf{R}}}{4\pi R^2} = \frac{I d\mathbf{l} \times \mathbf{R}}{4\pi R^3}.$$

Magnetic flux density:

$$\mathbf{B} = \mu_0 \mathbf{H}.$$

7.2 Straight wire along the z -axis

Example: derive $\mathbf{H}$ of an infinitely long straight wire

For a wire on the z -axis:

$$d\mathbf{l} = dz \hat{\mathbf{a}}_z.$$

Observation point:

$$P(\rho, \phi, z).$$

Use a source point z' on the wire:

$$\mathbf{R} = \rho \hat{\mathbf{a}}_\rho - z' \hat{\mathbf{a}}_z, \quad R = \sqrt{\rho^2 + z'^2}.$$

Then

$$d\mathbf{H} = \frac{I(dz' \hat{\mathbf{a}}_z) \times (\rho \hat{\mathbf{a}}_\rho - z' \hat{\mathbf{a}}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}.$$

Because

$$\hat{\mathbf{a}}_z \times \hat{\mathbf{a}}_\rho = \hat{\mathbf{a}}_\phi, \quad \hat{\mathbf{a}}_z \times \hat{\mathbf{a}}_z = 0,$$

$$d\mathbf{H} = \frac{I \rho dz'}{4\pi(\rho^2 + z'^2)^{3/2}} \hat{\mathbf{a}}_\phi.$$

Integrate:

$$\mathbf{H} = \frac{I \rho}{4\pi} \int_{-\infty}^{\infty} \frac{dz'}{(\rho^2 + z'^2)^{3/2}} \hat{\mathbf{a}}_\phi.$$

Use

$$\int_{-\infty}^{\infty} \frac{dz'}{(\rho^2 + z'^2)^{3/2}} = \frac{2}{\rho^2}.$$

5 Annular Sheet and Point-Charge Potential

5.1 Electric field on the axis of an annular sheet

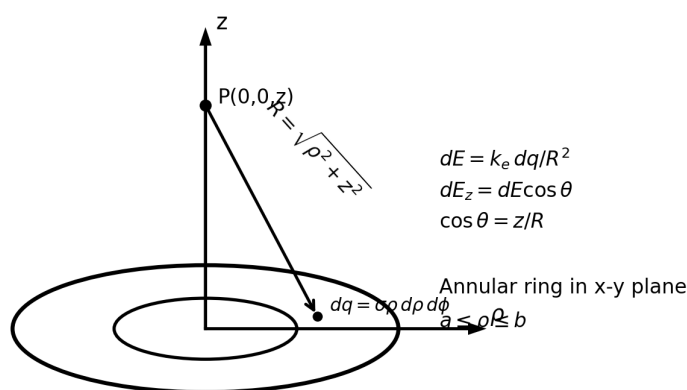


Figure 5.1: Clean annular sheet geometry for deriving the electric field on the z -axis.

Concept

For a uniform surface charge density σ on

$$z = 0, \quad a \leq \rho \leq b, \quad 0 \leq \phi \leq 2\pi,$$

use

$$dS = \rho d\rho d\phi, \quad dq = \sigma \rho d\rho d\phi.$$

For point $P(0, 0, z)$:

$$R = \sqrt{\rho^2 + z^2}, \quad \cos \theta = \frac{z}{R}.$$

Example: annular sheet axial field

Differential field

$$dE = k_e \frac{dq}{R^2},$$

$$dE_z = dE \cos \theta = k_e \frac{dq}{R^2} \left(\frac{z}{R} \right) = k_e \frac{z dq}{R^3}.$$

Substitute $dq = \sigma \rho d\rho d\phi$:

$$dE_z = k_e \frac{\sigma z \rho d\rho d\phi}{(\rho^2 + z^2)^{3/2}}.$$

Integrate

$$E_z = k_e \sigma z \int_0^{2\pi} \int_a^b \frac{\rho d\rho d\phi}{(\rho^2 + z^2)^{3/2}}.$$

Use

$$\int \frac{\rho d\rho}{(\rho^2 + z^2)^{3/2}} = -\frac{1}{\sqrt{\rho^2 + z^2}}.$$

Thus

$$E_z = 2\pi k_e \sigma z \left[\frac{1}{\sqrt{a^2 + z^2}} - \frac{1}{\sqrt{b^2 + z^2}} \right].$$

Since

$$2\pi k_e = \frac{1}{2\epsilon_0},$$

$$\mathbf{E} = \frac{\sigma z}{2\epsilon_0} \left[\frac{1}{\sqrt{a^2 + z^2}} - \frac{1}{\sqrt{b^2 + z^2}} \right] \hat{\mathbf{a}}_z$$

Plane limit:

$$a \rightarrow 0, \quad b \rightarrow \infty.$$

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0} \hat{\mathbf{a}}_z \quad (z > 0), \quad \mathbf{E} = -\frac{\sigma}{2\epsilon_0} \hat{\mathbf{a}}_z \quad (z < 0)$$

5.2 Point-charge potential

Concept

For a point charge Q at the origin:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{\mathbf{a}}_r.$$

Potential relative to infinity:

$$V(r) = - \int_{\infty}^r \mathbf{E} \cdot d\mathbf{l}.$$

Since

$$d\mathbf{l} = dr \hat{\mathbf{a}}_r,$$

then

$$\mathbf{E} \cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} dr.$$

Example: derive point-charge potential

$$V(r) = - \int_{\infty}^r \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} dr$$

$$= -\frac{Q}{4\pi\epsilon_0} \int_{\infty}^r r^{-2} dr.$$

Use

$$\int r^{-2} dr = -\frac{1}{r}.$$

Therefore

$$\begin{aligned} V(r) &= -\frac{Q}{4\pi\epsilon_0} \left[-\frac{1}{r} \right]_{\infty}^r \\ &= -\frac{Q}{4\pi\epsilon_0} \left[-\frac{1}{r} - 0 \right]. \end{aligned}$$

$$V(r) = \frac{Q}{4\pi\epsilon_0 r}$$

5.3 Electric field from a potential in cylindrical coordinates

Example: potential field at point P

Given

$$V = \frac{100\rho \cos \phi}{z^2 + 1}, \quad P : \rho = 3 \text{ m}, \quad \phi = 60^\circ, \quad z = 2 \text{ m}.$$

Potential

$$\begin{aligned} V(P) &= \frac{100(3) \cos 60^\circ}{2^2 + 1} \\ &= \frac{150}{5} = 30 \text{ V}. \end{aligned}$$

$$V(P) = 30 \text{ V}$$

Field formula

$$\mathbf{E} = -\nabla V = -\left(\frac{\partial V}{\partial \rho} \hat{\mathbf{a}}_\rho + \frac{1}{\rho} \frac{\partial V}{\partial \phi} \hat{\mathbf{a}}_\phi + \frac{\partial V}{\partial z} \hat{\mathbf{a}}_z \right).$$

Components:

$$\begin{aligned} E_\rho &= -\frac{\partial V}{\partial \rho} = -\frac{100 \cos \phi}{z^2 + 1}, \\ E_\phi &= -\frac{1}{\rho} \frac{\partial V}{\partial \phi} = -\frac{100 \sin \phi}{z^2 + 1}, \\ E_z &= -\frac{\partial V}{\partial z} = -\frac{200\rho z \cos \phi}{(z^2 + 1)^2}. \end{aligned}$$

At P:

$$\begin{aligned} E_\rho &= -\frac{100 \cos 60^\circ}{5} = -10, \\ E_\phi &= -\frac{100 \sin 60^\circ}{5} = -17.32, \\ E_z &= -\frac{200(3)(2) \cos 60^\circ}{25} = -24. \end{aligned}$$

$$\mathbf{E} = -10\hat{\mathbf{a}}_\rho + 17.32\hat{\mathbf{a}}_\phi + 24\hat{\mathbf{a}}_z \text{ V/m}$$

Chapter 4 Magnetostatics

Source pages: Document 204159 pp. 21, 24; Document 204315 pp. 15, 18; Document 204511 pp. 5, 7

4.1 Biot-Savart law for a straight filament

The handwritten notes use $\mathbf{H}$ form:

$$d\mathbf{H}(\mathbf{r}) = \frac{I d\mathbf{l} \times \hat{\mathbf{R}}}{4\pi R^2}$$

Equivalent form:

$$d\mathbf{H}(\mathbf{r}) = \frac{I d\mathbf{l} \times \mathbf{R}}{4\pi R^3}$$

For a wire along the z -axis:

$$d\mathbf{l} = dz \hat{\mathbf{a}}_z$$

$$\mathbf{R} = \rho \hat{\mathbf{a}}_\rho - z \hat{\mathbf{a}}_z$$

$$R = \sqrt{\rho^2 + z^2}$$

Cross products:

$$\hat{\mathbf{a}}_z \times \hat{\mathbf{a}}_z = 0, \quad \hat{\mathbf{a}}_z \times \hat{\mathbf{a}}_\rho = \hat{\mathbf{a}}_\phi$$

Then

$$d\mathbf{H} = \frac{I(dz \hat{\mathbf{a}}_z) \times (\rho \hat{\mathbf{a}}_\rho - z \hat{\mathbf{a}}_z)}{4\pi(\rho^2 + z^2)^{3/2}}$$

$$d\mathbf{H} = \frac{I \rho dz \hat{\mathbf{a}}_\phi}{4\pi(\rho^2 + z^2)^{3/2}}$$

Integrate:

$$\mathbf{H} = \int_{-\infty}^{\infty} d\mathbf{H} = \frac{I \rho \hat{\mathbf{a}}_\phi}{4\pi} \int_{-\infty}^{\infty} \frac{dz}{(\rho^2 + z^2)^{3/2}}$$

Using

$$\int \frac{dz}{(\rho^2 + z^2)^{3/2}} = \frac{z}{\rho^2 \sqrt{\rho^2 + z^2}},$$

we obtain

$$\mathbf{H} = \frac{I \rho \hat{\mathbf{a}}_\phi}{4\pi} \left[\frac{z}{\rho^2 \sqrt{\rho^2 + z^2}} \right]_{-\infty}^{\infty}$$

$$\mathbf{H} = \frac{I \rho \hat{\mathbf{a}}_\phi}{4\pi} \left(\frac{1}{\rho^2} - \left[-\frac{1}{\rho^2} \right] \right)$$

$$\mathbf{H} = \frac{I}{2\pi \rho} \hat{\mathbf{a}}_\phi$$

The notes also write the equivalent current relation:

$$I = 2\pi \rho H_\phi$$

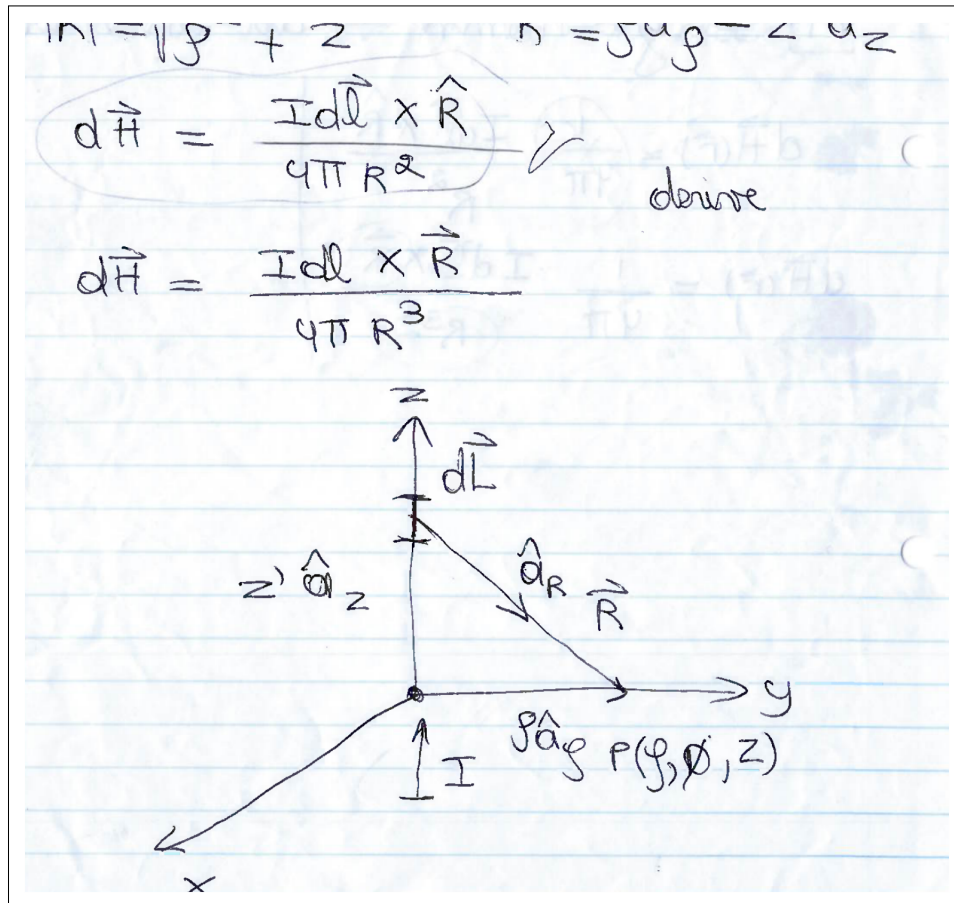


Figure 7: Original Biot-Savart straight-filament geometry reused from the handwritten notes.

Final:

$$\mathbf{H} = \frac{I}{2\pi\rho} \hat{\mathbf{a}}_\phi$$

$$\mathbf{B} = \frac{\mu_0 I}{2\pi\rho} \hat{\mathbf{a}}_\phi$$

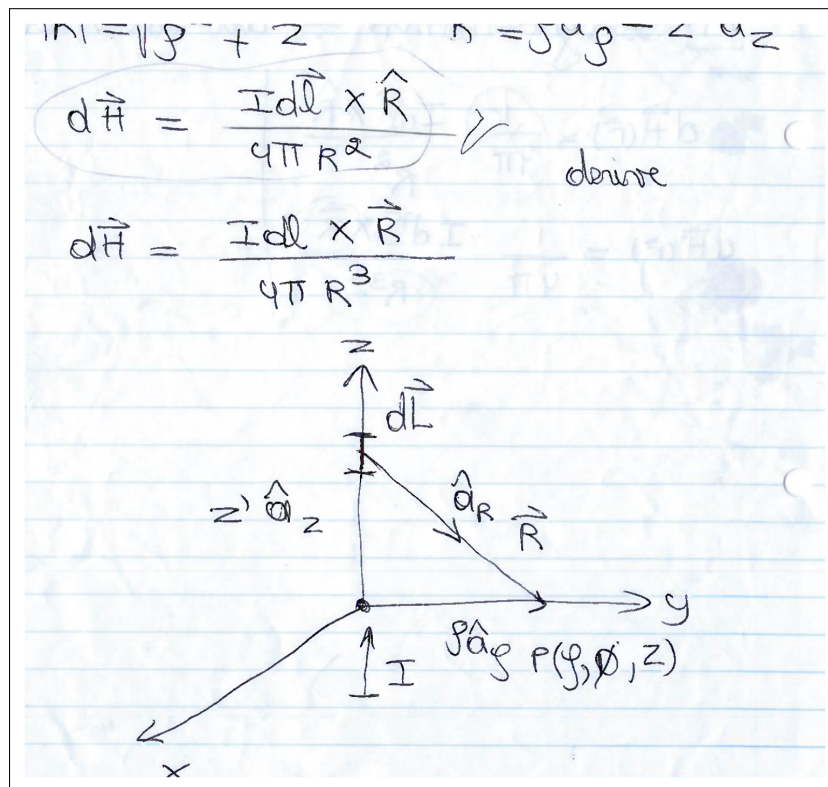


Figure 7: Original Biot-Savart straight-wire sketch reused from the concept notes.

7.3 Ampere law for a long wire

Example: use Ampere law for the same wire

Ampere law:

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I_{\text{enc.}}$$

Choose circular path of radius ρ :

$$d\mathbf{l} = \rho d\phi \hat{\mathbf{a}}_\phi.$$

Since $\mathbf{H} = H_\phi \hat{\mathbf{a}}_\phi$ is constant on the circular path:

$$H_\phi \int_0^{2\pi} \rho d\phi = I.$$

$$H_\phi(2\pi\rho) = I.$$

$$\mathbf{H} = \frac{I}{2\pi\rho} \hat{\mathbf{a}}_\phi$$

7.4 Circular current loop on the axis

Example: circular loop of radius a in the xy -plane

At point $P(0, 0, z)$ on the axis:

$$R = \sqrt{a^2 + z^2}.$$

Use

$$d\mathbf{l} = a d\phi \hat{\mathbf{a}}_\phi, \quad \mathbf{R} = -a\hat{\mathbf{a}}_\rho + z\hat{\mathbf{a}}_z.$$

Then

$$d\mathbf{H} = \frac{I d\mathbf{l} \times \mathbf{R}}{4\pi R^3}.$$

$$d\mathbf{l} \times \mathbf{R} = a d\phi \hat{\mathbf{a}}_\phi \times (-a\hat{\mathbf{a}}_\rho + z\hat{\mathbf{a}}_z).$$

Use

$$\hat{\mathbf{a}}_\phi \times \hat{\mathbf{a}}_\rho = -\hat{\mathbf{a}}_z, \quad \hat{\mathbf{a}}_\phi \times \hat{\mathbf{a}}_z = \hat{\mathbf{a}}_\rho.$$

Thus

$$d\mathbf{H} = \frac{I}{4\pi R^3} (a^2 d\phi \hat{\mathbf{a}}_z + az d\phi \hat{\mathbf{a}}_\rho).$$

By symmetry, the $\hat{\mathbf{a}}_\rho$ components cancel around the loop. Therefore

$$\mathbf{H} = \int_0^{2\pi} \frac{Ia^2}{4\pi(a^2 + z^2)^{3/2}} d\phi \hat{\mathbf{a}}_z.$$

$$\mathbf{H} = \frac{Ia^2}{2(a^2 + z^2)^{3/2}} \hat{\mathbf{a}}_z$$

$$\mathbf{B} = \frac{\mu_0 Ia^2}{2(a^2 + z^2)^{3/2}} \hat{\mathbf{a}}_z$$

7.5 Line integral of a given magnetic field

Example: evaluate $\oint_C \mathbf{B} \cdot d\mathbf{l}$

Given

$$\mathbf{B} = \frac{\mu_0}{2\pi\rho} \hat{\mathbf{a}}_\phi.$$

Circle C in the xy -plane with radius $\rho = 0.1$ m:

$$d\mathbf{l} = \rho d\phi \hat{\mathbf{a}}_\phi.$$

Then

$$\mathbf{B} \cdot d\mathbf{l} = \frac{\mu_0}{2\pi\rho} \hat{\mathbf{a}}_\phi \cdot (\rho d\phi \hat{\mathbf{a}}_\phi) = \frac{\mu_0}{2\pi} d\phi.$$

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \int_0^{2\pi} \frac{\mu_0}{2\pi} d\phi = \mu_0.$$

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0$$

Chapter 8 Waves and Optics

6 Magnetostatics: Biot-Savart Law and Ampere Law

6.1 Biot-Savart law

Concept

Magnetic field intensity form:

$$d\mathbf{H} = \frac{I d\mathbf{l} \times \hat{\mathbf{R}}}{4\pi R^2} = \frac{I d\mathbf{l} \times \mathbf{R}}{4\pi R^3}.$$

Magnetic flux density:

$$\mathbf{B} = \mu_0 \mathbf{H}.$$

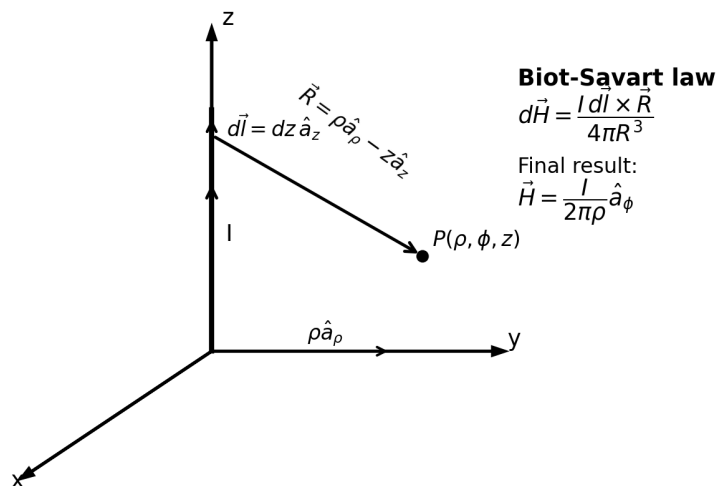


Figure 6.1: Clean Biot-Savart straight-wire geometry.

Example: straight wire along the z-axis

For an infinitely long wire on the z -axis:

$$d\mathbf{l} = dz' \hat{\mathbf{a}}_z.$$

Observation point:

$$P(\rho, \phi, z).$$

Use

$$\mathbf{R} = \rho \hat{\mathbf{a}}_\rho - z' \hat{\mathbf{a}}_z, \quad R = \sqrt{\rho^2 + z'^2}.$$

Then

$$d\mathbf{H} = \frac{I(dz'\hat{\mathbf{a}}_z) \times (\rho\hat{\mathbf{a}}_\rho - z'\hat{\mathbf{a}}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}.$$

Because

$$\hat{\mathbf{a}}_z \times \hat{\mathbf{a}}_\rho = \hat{\mathbf{a}}_\phi, \quad \hat{\mathbf{a}}_z \times \hat{\mathbf{a}}_z = 0,$$

we get

$$d\mathbf{H} = \frac{I\rho dz'}{4\pi(\rho^2 + z'^2)^{3/2}}\hat{\mathbf{a}}_\phi.$$

Integrate:

$$\mathbf{H} = \frac{I\rho}{4\pi} \int_{-\infty}^{\infty} \frac{dz'}{(\rho^2 + z'^2)^{3/2}} \hat{\mathbf{a}}_\phi.$$

Use

$$\int_{-\infty}^{\infty} \frac{dz'}{(\rho^2 + z'^2)^{3/2}} = \frac{2}{\rho^2}.$$

$$\mathbf{H} = \frac{I}{2\pi\rho} \hat{\mathbf{a}}_\phi$$

$$\mathbf{B} = \frac{\mu_0 I}{2\pi\rho} \hat{\mathbf{a}}_\phi$$

6.2 Ampere law for the same wire

Example: long wire by Ampere law

Ampere law:

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I_{\text{enc.}}$$

Choose a circular path of radius ρ :

$$d\mathbf{l} = \rho d\phi \hat{\mathbf{a}}_\phi.$$

Since $\mathbf{H} = H_\phi \hat{\mathbf{a}}_\phi$ is constant along the circular path:

$$\begin{aligned} \oint_C \mathbf{H} \cdot d\mathbf{l} &= H_\phi \int_0^{2\pi} \rho d\phi \\ &= H_\phi (2\pi\rho) = I. \end{aligned}$$

$$\mathbf{H} = \frac{I}{2\pi\rho} \hat{\mathbf{a}}_\phi$$

6.3 Line integral of a given magnetic field

Example: magnetic line integral

Given

$$\mathbf{B} = \frac{\mu_0}{2\pi\rho} \hat{\mathbf{a}}_\phi.$$

For a circle in the xy -plane:

$$d\mathbf{l} = \rho d\phi \hat{\mathbf{a}}_\phi.$$

Then

$$\begin{aligned}\mathbf{B} \cdot d\mathbf{l} &= \frac{\mu_0}{2\pi\rho} \hat{\mathbf{a}}_\phi \cdot (\rho d\phi \hat{\mathbf{a}}_\phi) \\ &= \frac{\mu_0}{2\pi} d\phi.\end{aligned}$$

Thus

$$\begin{aligned}\oint_C \mathbf{B} \cdot d\mathbf{l} &= \int_0^{2\pi} \frac{\mu_0}{2\pi} d\phi \\ &= \frac{\mu_0}{2\pi} [\phi]_0^{2\pi}.\end{aligned}$$

$$\boxed{\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0}$$

Chapter 6 Waves, Optics, and Radiation

Source pages: Document 204159 pp. 14, 15, 22, 25; Document 204315 pp. 8, 9, 16, 19; Document 204511 pp. 4, 6, 10, 13, 14

6.1 Double-slit small-angle relations

For small angles:

$$\sin \theta \approx \tan \theta \approx \frac{y}{x}$$

Bright fringes:

$$d \sin \theta = m\lambda$$

$$d \left(\frac{y}{x} \right) = m\lambda$$

$$y_m = \frac{xm\lambda}{d}$$

Next bright fringe:

$$y_{m+1} = \frac{x(m+1)\lambda}{d}$$

Fringe spacing:

$$\Delta y = y_{m+1} - y_m$$

$$\Delta y = \frac{x(m+1)\lambda}{d} - \frac{xm\lambda}{d}$$

$$\Delta y = \frac{xm\lambda + x\lambda - xm\lambda}{d}$$

$$\Delta y = \frac{x\lambda}{d}$$

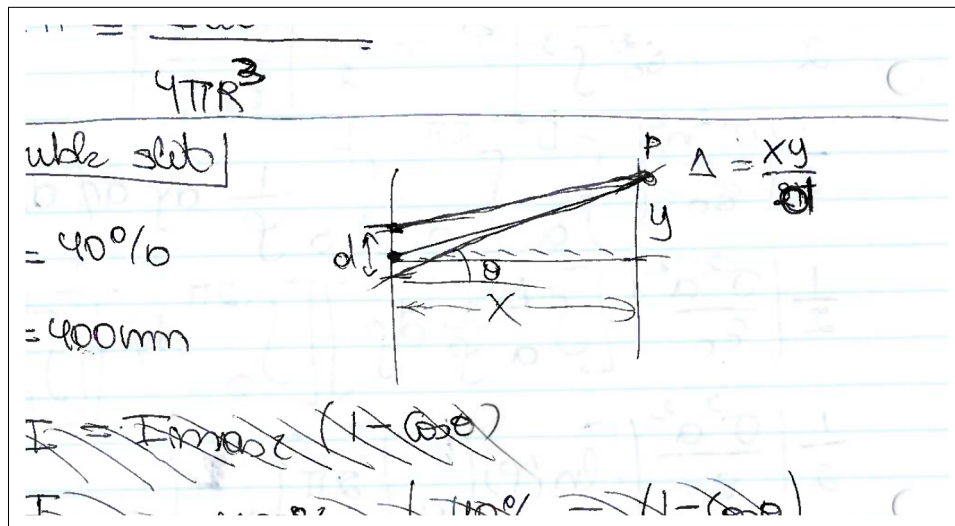


Figure 9: Original double-slit geometry sketch reused from the handwritten notes.

6.2 Double-slit intensity, phase, and path difference

Given

$$\frac{I}{I_{\max}} = 40\% = \frac{40}{100} = 0.40$$

For double slit:

$$\frac{I}{I_{\max}} = \cos^2 \left(\frac{\phi}{2} \right)$$

Therefore

$$0.40 = \cos^2 \left(\frac{\phi}{2} \right)$$

$$\cos \left(\frac{\phi}{2} \right) = \sqrt{0.40}$$

$$\frac{\phi}{2} = \cos^{-1}(\sqrt{0.40})$$

$$\phi = 2 \cos^{-1}(\sqrt{0.40}) = 1.772154 \text{ rad}$$

Phase-path relation:

$$\phi = \frac{2\pi}{\lambda} \delta$$

$$\delta = \frac{\phi \lambda}{2\pi}$$

For $\lambda = 400 \text{ nm}$:

$$\delta = \frac{(1.772154)(400 \text{ nm})}{2\pi}$$

$$\delta \approx 113 \text{ nm}$$

6.3 Compton scattering

Compton shift:

$$\Delta\lambda = \frac{h}{m_e c} (1 - \cos \theta)$$

Compton wavelength of the electron:

$$\frac{h}{m_e c} = \frac{6.626070 \times 10^{-34} \text{ J} \cdot \text{s}}{(9.11 \times 10^{-31} \text{ kg})(3.00 \times 10^8 \text{ m/s})}$$

$$\frac{h}{m_e c} = 0.00243 \text{ nm}$$

Scattered wavelength:

$$\Delta\lambda = \lambda' - \lambda$$

$$\lambda' - \lambda = \frac{h}{m_e c} (1 - \cos \theta)$$

$$\lambda' = \lambda + \frac{h}{m_e c} (1 - \cos \theta)$$

Example result from the notes:

$$\lambda' = 0.271 \text{ nm}$$

Energy of the scattered photon:

$$E' = \frac{hc}{\lambda'}$$

Using the convenient constant:

$$E' = \frac{1240 \text{ eV} \cdot \text{nm}}{0.271 \text{ nm}}$$

$$E' = 4.57 \text{ keV}$$

Kinetic energy of the scattered electron:

$$K_e = E_i - E'$$

where

$$E_i = \frac{hc}{\lambda}.$$

6.4 Brewster angle and critical angle

Brewster angle:

$$\tan \theta_B = \frac{n_2}{n_1}$$

For air to glass:

$$n_{\text{air}} = 1, \quad n_{\text{glass}} = 1.5$$

$$\tan \theta_B = \frac{n_{\text{glass}}}{n_{\text{air}}} = \frac{1.5}{1}$$

$$\theta_B = \tan^{-1} \left(\frac{n_{\text{glass}}}{n_{\text{air}}} \right)$$

Critical angle:

$$\theta_c = \sin^{-1} \left(\frac{n_2}{n_1} \right)$$

This applies when light travels from the higher-index medium to the lower-index medium.

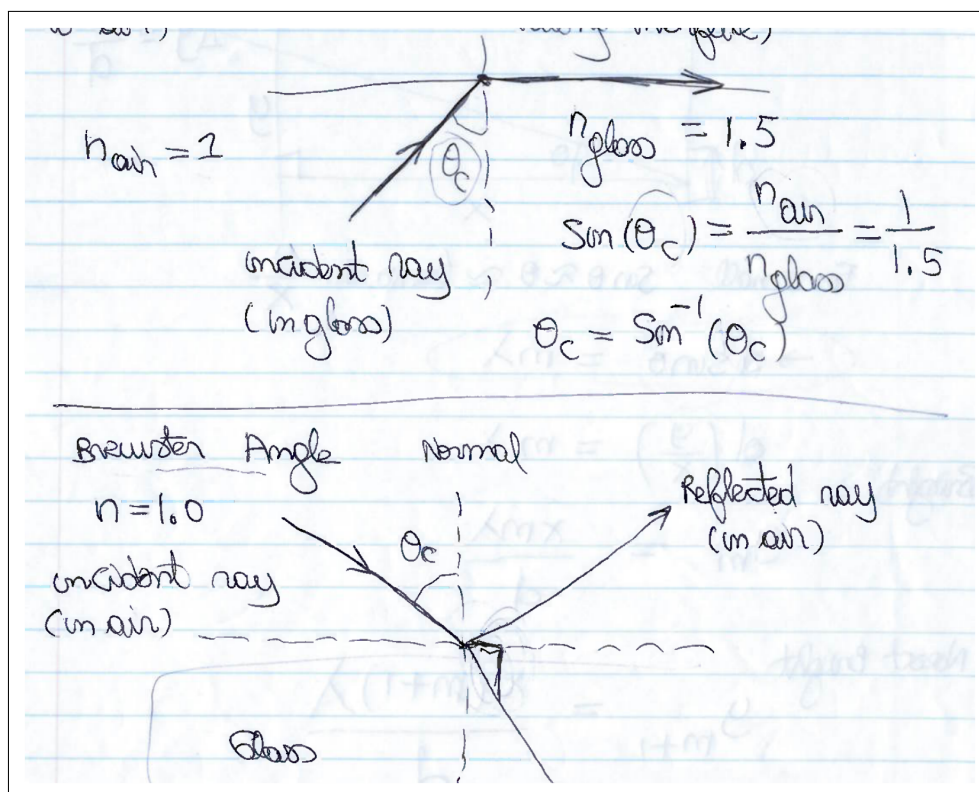


Figure 10: Original Brewster-angle and refraction sketch reused from the handwritten notes.

6.5 Stefan-Boltzmann law and Wien's law

Stefan-Boltzmann law:

$$P = \sigma AT^4$$

For a spherical star:

$$A = 4\pi R^2$$

Thus

$$P = \sigma(4\pi R^2)T^4$$

Solve for temperature:

$$T^4 = \frac{P}{\sigma(4\pi R^2)}$$

$$T = \left(\frac{P}{\sigma(4\pi R^2)} \right)^{1/4}$$

Wien's displacement law:

$$\lambda_{\max} T = 2.898 \times 10^{-3} \text{ m} \cdot \text{K}$$

$$\lambda_{\max} = \frac{2.898 \times 10^{-3}}{T}$$

Example notes:

$$P = 4.5 \times 10^{30} \text{ W}, R = 8.2 \times 10^9 \text{ m}$$

$$T = 1.75 \times 10^4 \text{ K}$$

$$\lambda_{\max} = 2.898 \times 10^{-3} \times \frac{1}{1.75 \times 10^4}$$

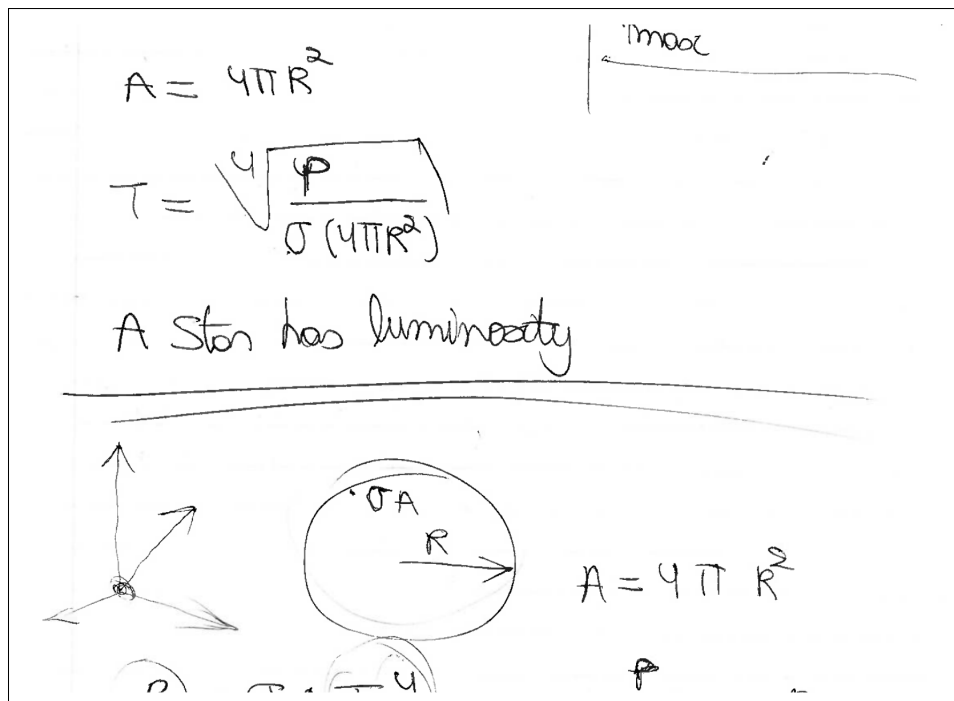


Figure 11: Original star-luminosity sketch reused from the handwritten notes.

8.1 Maxwell wave equation in free space

Concept

Free space Maxwell equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

Also

$$\nabla \cdot \mathbf{E} = 0.$$

Example: derive $\nabla^2 \mathbf{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$

Start with Faraday law:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

Take curl of both sides:

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}).$$

Substitute Ampere-Maxwell law:

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

Use vector identity:

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}.$$

Since $\nabla \cdot \mathbf{E} = 0$:

$$-\nabla^2 \mathbf{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

$$\boxed{\nabla^2 \mathbf{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}}$$

8.2 Critical angle and Brewster angle

Concept

Critical angle for light going from high index n_1 to low index n_2 :

$$\theta_c = \sin^{-1} \left(\frac{n_2}{n_1} \right).$$

Brewster angle:

$$\tan \theta_B = \frac{n_2}{n_1}.$$

Example: air and glass

For air to glass:

$$n_{\text{air}} = 1, \quad n_{\text{glass}} = 1.5.$$

Critical angle exists when the ray travels from glass to air:

$$\theta_c = \sin^{-1} \left(\frac{n_{\text{air}}}{n_{\text{glass}}} \right) = \sin^{-1} \left(\frac{1}{1.5} \right).$$

$$\boxed{\theta_c = 41.8^\circ}$$

Brewster angle for incidence from air onto glass:

$$\theta_B = \tan^{-1} \left(\frac{n_{\text{glass}}}{n_{\text{air}}} \right) = \tan^{-1}(1.5).$$

$$\theta_B = 56.3^\circ$$

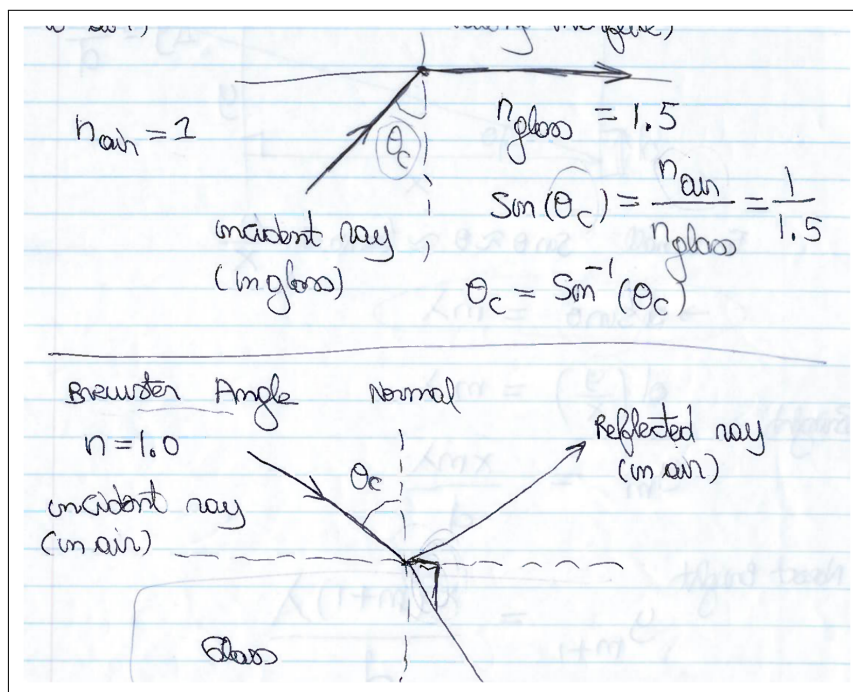


Figure 8: Original Brewster-angle/refraction sketch reused from the concept notes.

8.3 Double-slit fringe spacing

Concept

Bright fringes:

$$d \sin \theta = m\lambda.$$

Small angle:

$$\sin \theta \approx \tan \theta \approx \frac{y}{x}.$$

Therefore

$$y_m = \frac{xm\lambda}{d}.$$

Example: derive fringe spacing

For next bright fringe:

$$y_{m+1} = \frac{x(m+1)\lambda}{d}.$$

Spacing between adjacent bright fringes:

$$\Delta y = y_{m+1} - y_m.$$

$$\Delta y = \frac{x(m+1)\lambda}{d} - \frac{xm\lambda}{d}.$$

$$\Delta y = \frac{x\lambda}{d} [(m+1) - m].$$

$$\Delta y = \frac{x\lambda}{d}$$

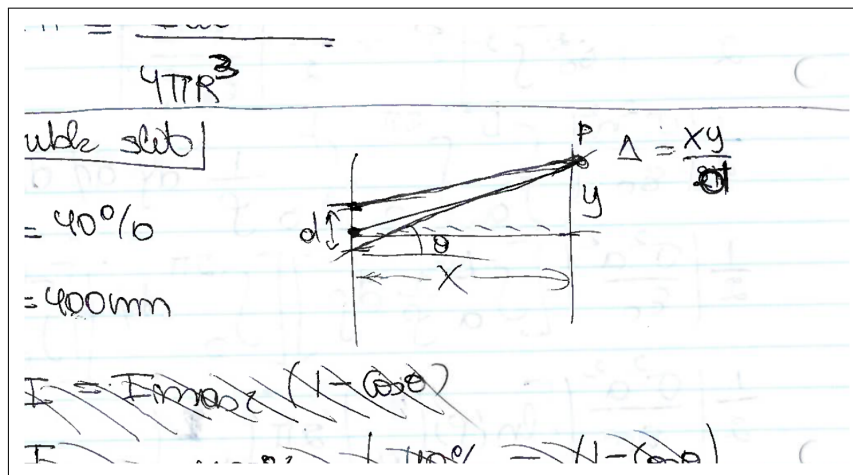


Figure 9: Original double-slit geometry sketch reused from the concept notes.

8.4 Double-slit intensity, phase, and path difference

Concept

Intensity formula:

$$I = I_{\max} \cos^2 \left(\frac{\phi}{2} \right).$$

Phase-path relation:

$$\phi = \frac{2\pi}{\lambda} \Delta.$$

Example: intensity is 40% of maximum

Given

$$\frac{I}{I_{\max}} = 0.40.$$

$$0.40 = \cos^2 \left(\frac{\phi}{2} \right).$$

$$\cos \left(\frac{\phi}{2} \right) = \sqrt{0.40}.$$

$$\frac{\phi}{2} = \cos^{-1}(\sqrt{0.40}).$$

$$\phi = 2 \cos^{-1}(\sqrt{0.40}) = 1.77 \text{ rad}$$

For $\lambda = 400 \text{ nm}$:

$$\Delta = \frac{\phi \lambda}{2\pi} = \frac{(1.772)(400 \text{ nm})}{2\pi}.$$

$$\Delta = 113 \text{ nm}$$

8.5 Diffraction grating

Concept

Principal maxima occur when

$$d \sin \theta = n\lambda.$$

Resolving power:

$$RP = \frac{\lambda}{\Delta\lambda} = Nn.$$

Example: sodium D lines

Resolve

$$\lambda_1 = 589.0 \text{ nm}, \quad \lambda_2 = 589.6 \text{ nm}.$$

Mean wavelength:

$$\lambda \approx \frac{589.0 + 589.6}{2} = 589.3 \text{ nm}.$$

Difference:

$$\Delta\lambda = 0.6 \text{ nm}.$$

First order:

$$n = 1.$$

$$RP = \frac{\lambda}{\Delta\lambda} = Nn.$$

$$N = \frac{\lambda}{n\Delta\lambda} = \frac{589.3}{1(0.6)} = 982.2.$$

$$N \approx 983 \text{ slits/rulings}$$

Chapter 9 Black-Body Radiation and Photoelectric Effect

9.1 Stefan-Boltzmann and Wien laws

Concept

Stefan-Boltzmann law:

$$P = \sigma AT^4.$$

For a star:

$$A = 4\pi R^2.$$

Therefore

$$T = \left(\frac{P}{\sigma 4\pi R^2} \right)^{1/4}$$

Wien displacement law:

$$\lambda_{\max} T = 2.898 \times 10^{-3} \text{ m} \cdot \text{K}$$

Example: surface temperature and peak wavelength of the Sun

Given

$$R = 6.96 \times 10^8 \text{ m}, \quad P = 3.85 \times 10^{26} \text{ W},$$

$$\sigma = 5.670390 \times 10^{-8} \text{ W/m}^2\text{K}^4.$$

$$T = \left(\frac{3.85 \times 10^{26}}{(5.670390 \times 10^{-8})(4\pi)(6.96 \times 10^8)^2} \right)^{1/4}.$$

$$T = 5.78 \times 10^3 \text{ K}$$

$$\lambda_{\text{max}} = \frac{2.898 \times 10^{-3}}{5.779 \times 10^3}.$$

$$\lambda_{\text{max}} = 5.01 \times 10^{-7} \text{ m} = 501 \text{ nm}$$

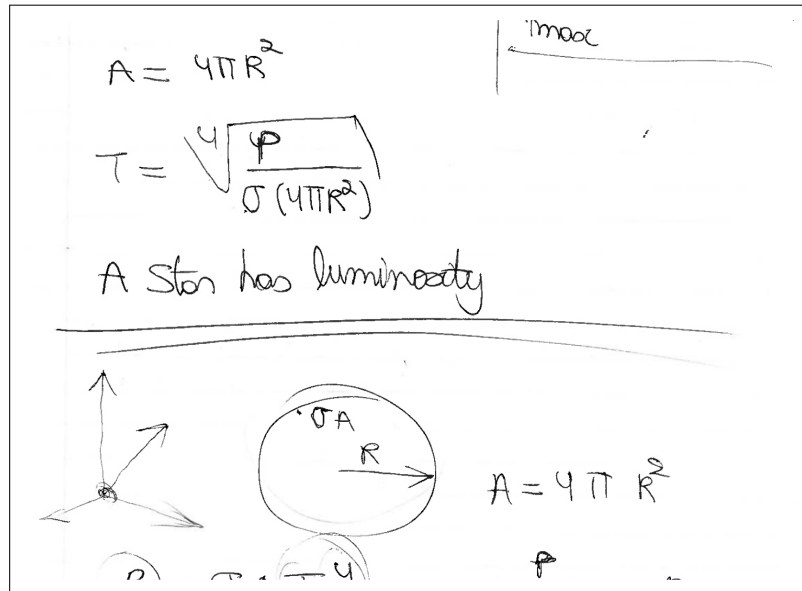


Figure 10: Original star luminosity / black-body sketch reused from the concept notes.

9.2 Photoelectric effect

Concept

Photon energy:

$$E_{\text{photon}} = hf = \frac{hc}{\lambda}.$$

In electron-volts using $hc = 1240 \text{ eV} \cdot \text{nm}$:

$$E_{\text{photon}} = \frac{1240}{\lambda(\text{nm})} \text{ eV}.$$

Einstein photoelectric equation:

$$K_{\text{max}} = E_{\text{photon}} - \phi.$$

Example: two metals illuminated by the same light

Metal A:

$$\phi_A = 4.7 \text{ eV}, \quad K_A = 0.8 \text{ eV}.$$

Photon energy:

$$E_{\text{photon}} = K_A + \phi_A = 0.8 + 4.7 = 5.5 \text{ eV}.$$

Wavelength:

$$\lambda = \frac{1240 \text{ eV} \cdot \text{nm}}{5.5 \text{ eV}}.$$

$$\lambda = 225 \text{ nm}$$

Metal B:

$$K_B = 3.2 \text{ eV}.$$

7 Waves and Optics

7.1 Critical angle and Brewster angle

Concept

Critical angle, for light going from high refractive index n_1 to low refractive index n_2 :

$$\theta_c = \sin^{-1} \left(\frac{n_2}{n_1} \right).$$

Brewster angle:

$$\tan \theta_B = \frac{n_2}{n_1}.$$

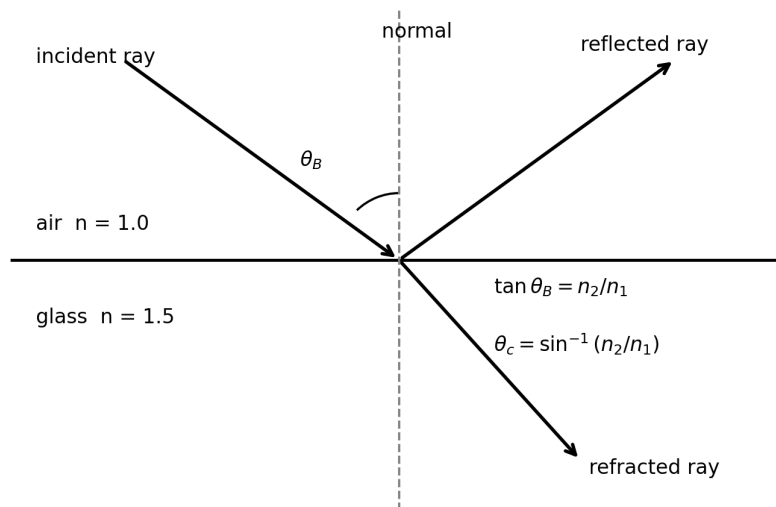


Figure 7.1: Clean ray diagram for critical angle and Brewster angle.

Example: air and glass

For air and glass:

$$n_{\text{air}} = 1.0, \quad n_{\text{glass}} = 1.5.$$

Critical angle exists for glass to air:

$$\begin{aligned} \theta_c &= \sin^{-1} \left(\frac{n_{\text{air}}}{n_{\text{glass}}} \right) \\ &= \sin^{-1} \left(\frac{1}{1.5} \right). \end{aligned}$$

$$\theta_c = 41.8^\circ$$

Brewster angle for air to glass:

$$\begin{aligned}\theta_B &= \tan^{-1} \left(\frac{n_{\text{glass}}}{n_{\text{air}}} \right) \\ &= \tan^{-1}(1.5).\end{aligned}$$

$$\theta_B = 56.3^\circ$$

7.2 Double-slit fringe spacing

Concept

Bright fringes:

$$d \sin \theta = m\lambda.$$

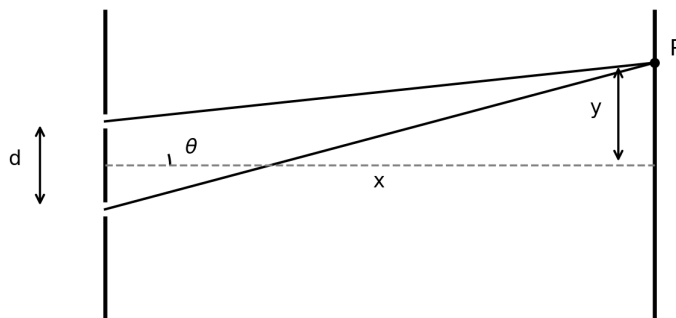
Small-angle approximation:

$$\sin \theta \approx \tan \theta \approx \frac{y}{x}.$$

Therefore:

$$y_m = \frac{xm\lambda}{d}.$$

Double-slit geometry



Small angle: $\sin \theta \approx \tan \theta \approx y/x$

Bright fringes: $d \sin \theta = m\lambda$

$$\Delta y = x\lambda/d$$

Figure 7.2: Clean double-slit geometry for deriving fringe spacing.

Example: derive fringe spacing

For the next bright fringe:

$$y_{m+1} = \frac{x(m+1)\lambda}{d}.$$

Spacing between adjacent bright fringes:

$$\begin{aligned}\Delta y &= y_{m+1} - y_m \\ &= \frac{x(m+1)\lambda}{d} - \frac{xm\lambda}{d} \\ &= \frac{x\lambda}{d}[(m+1) - m].\end{aligned}$$

$$\Delta y = \frac{x\lambda}{d}$$

7.3 Double-slit intensity, phase, and path difference

Concept

Intensity relation:

$$I = I_{\max} \cos^2 \left(\frac{\phi}{2} \right).$$

Phase-path relation:

$$\phi = \frac{2\pi}{\lambda} \Delta.$$

Example: intensity is 40 percent of maximum

Given

$$\frac{I}{I_{\max}} = 0.40.$$

Then

$$\begin{aligned}0.40 &= \cos^2 \left(\frac{\phi}{2} \right), \\ \cos \left(\frac{\phi}{2} \right) &= \sqrt{0.40}, \\ \frac{\phi}{2} &= \cos^{-1}(\sqrt{0.40}), \\ \phi &= 2 \cos^{-1}(\sqrt{0.40}).\end{aligned}$$

$$\phi = 1.77 \text{ rad}$$

For $\lambda = 400 \text{ nm}$:

$$\Delta = \frac{\phi\lambda}{2\pi} = \frac{(1.772)(400 \text{ nm})}{2\pi}.$$

$$\Delta = 113 \text{ nm}$$

8 Black-Body Radiation and Photoelectric Effect

8.1 Stefan-Boltzmann law and Wien law

Concept

Stefan-Boltzmann law:

$$P = \sigma AT^4.$$

For a spherical star:

$$A = 4\pi R^2.$$

Therefore:

$$T = \left(\frac{P}{\sigma 4\pi R^2} \right)^{1/4}.$$

Wien displacement law:

$$\lambda_{\max} T = 2.898 \times 10^{-3} \text{ m} \cdot \text{K}.$$

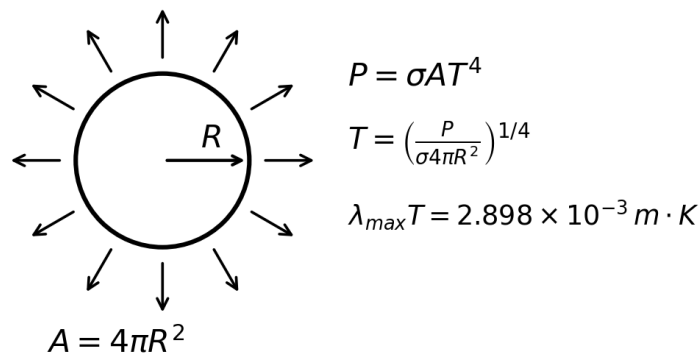


Figure 8.1: Clean star luminosity diagram for black-body radiation.

Example: blue giant star

Given

$$P = 4.5 \times 10^{30} \text{ W}, \quad R = 8.2 \times 10^9 \text{ m}, \quad \sigma = 5.67 \times 10^{-8} \text{ W/m}^2\text{K}^4.$$

Temperature

$$T = \left[\frac{4.5 \times 10^{30}}{(5.67 \times 10^{-8})(4\pi)(8.2 \times 10^9)^2} \right]^{1/4}.$$

$$T = 1.75 \times 10^4 \text{ K}$$

Peak wavelength

$$\lambda_{\max} = \frac{2.898 \times 10^{-3}}{1.75 \times 10^4}.$$

$$\lambda_{\max} = 1.66 \times 10^{-7} \text{ m} = 166 \text{ nm}$$

8.2 Photoelectric effect

Concept

Photon energy:

$$E_{\text{photon}} = hf = \frac{hc}{\lambda}.$$

Using $hc = 1240 \text{ eV nm}$:

$$E_{\text{photon}} = \frac{1240}{\lambda(\text{nm})} \text{ eV}.$$

Einstein photoelectric equation:

$$K_{\max} = E_{\text{photon}} - \phi.$$

Example: two metals illuminated by the same light

Metal A:

$$\phi_A = 4.2 \text{ eV}, \quad K_A = 1.2 \text{ eV}.$$

Photon energy:

$$\begin{aligned} E_{\text{photon}} &= K_A + \phi_A \\ &= 1.2 + 4.2 = 5.4 \text{ eV}. \end{aligned}$$

Wavelength:

$$\lambda = \frac{1240 \text{ eV nm}}{5.4 \text{ eV}}.$$

$$\lambda = 230 \text{ nm}$$

For Metal B:

$$K_B = 2.8 \text{ eV}.$$

Work function:

$$\begin{aligned} \phi_B &= E_{\text{photon}} - K_B \\ &= 5.4 - 2.8. \end{aligned}$$

$$\phi_B = 2.6 \text{ eV}$$

$$\phi_B = E_{\text{photon}} - K_B = 5.5 - 3.2.$$

$$\boxed{\phi_B = 2.3 \text{ eV}}$$

Chapter 10 Compton Scattering

10.1 Compton wavelength shift

Concept

$$\lambda' = \lambda + \frac{h}{m_e c} (1 - \cos \theta)$$

where

$$\frac{h}{m_e c} = 0.00243 \text{ nm}.$$

Photon energy:

$$E = \frac{hc}{\lambda} = \frac{1240 \text{ eV} \cdot \text{nm}}{\lambda(\text{nm})}.$$

Kinetic energy transferred to electron:

$$K_e = E_i - E'.$$

Example: $\lambda = 0.2700 \text{ nm}$ and $\theta = 65.0^\circ$

$$\lambda' = 0.2700 + 0.00243(1 - \cos 65^\circ) \text{ nm}.$$

$$\boxed{\lambda' = 0.271 \text{ nm}}$$

Scattered photon energy:

$$E' = \frac{1240}{0.2714} \text{ eV} = 4.57 \times 10^3 \text{ eV}.$$

$$\boxed{E' = 4.57 \text{ keV}}$$

Incident photon energy:

$$E_i = \frac{1240}{0.2700} \text{ eV} = 4.59 \text{ keV}.$$

Electron kinetic energy:

$$K_e = E_i - E' = 4.593 - 4.569 = 0.024 \text{ keV}.$$

$$\boxed{K_e \approx 24 \text{ eV}}$$

Chapter 11 Quantum Mechanics: Particle in a Box

11.1 Infinite square well

Concept

For

$$0 \leq x \leq L, \quad V(x) = 0,$$

Schrodinger equation becomes

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} = E\psi.$$

Rearrange:

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0, \quad k^2 = \frac{2mE}{\hbar^2}.$$

9 Compton Scattering

9.1 Compton wavelength shift

Concept

Compton shift:

$$\lambda' = \lambda + \frac{h}{m_e c} (1 - \cos \theta).$$

where

$$\frac{h}{m_e c} = 0.00243 \text{ nm}.$$

Photon energy:

$$E = \frac{hc}{\lambda} = \frac{1240 \text{ eV nm}}{\lambda(\text{ nm})}.$$

Kinetic energy transferred to electron:

$$K_e = E_i - E'.$$

Example: Compton scattering calculation

Scattered wavelength

$$\lambda' = 0.2700 + 0.00243(1 - \cos 65^\circ) \text{ nm}.$$

$$\lambda' = 0.271 \text{ nm}$$

Scattered photon energy

$$E' = \frac{1240}{0.2714} \text{ eV}.$$

$$E' = 4.57 \text{ keV}$$

Kinetic energy of electron

$$E_i = \frac{1240}{0.2700} \text{ eV} = 4.59 \text{ keV},$$
$$K_e = E_i - E' = 4.593 - 4.569 \text{ keV}.$$

$$K_e \approx 24 \text{ eV}$$

Chapter 5 Quantum Mechanics

Source pages: Document 204159 pp. 6, 12, 13, 16, 17, 27; Document 204315 pp. 6, 7, 10, 11, 21; Document 204511 pp. 1, 11, 12, 15, 16, 22

5.1 Time-dependent Schrodinger equation

General form:

$$\boxed{-\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x)\psi(x,t) = i\hbar \frac{\partial \psi(x,t)}{\partial t}}$$

For the free region where $V(x) = 0$:

$$-\frac{\hbar^2}{2m} [\psi(x,t)]''_x = i\hbar [\psi(x,t)]'_t$$

Use the trial wavefunction:

$$\psi(x,t) = A \sin(kx) \exp\left(-\frac{iEt}{\hbar}\right)$$

Spatial derivative:

$$\frac{\partial^2}{\partial x^2} \left[A \sin(kx) \exp\left(-\frac{iEt}{\hbar}\right) \right] = -k^2 A \sin(kx) \exp\left(-\frac{iEt}{\hbar}\right)$$

Time derivative:

$$\frac{\partial}{\partial t} \left[A \sin(kx) \exp\left(-\frac{iEt}{\hbar}\right) \right] = A \sin(kx) \left(-\frac{iE}{\hbar}\right) \exp\left(-\frac{iEt}{\hbar}\right)$$

Substitution gives

$$\frac{\hbar^2 k^2}{2m} A \sin(kx) \exp\left(-\frac{iEt}{\hbar}\right) = EA \sin(kx) \exp\left(-\frac{iEt}{\hbar}\right)$$

Therefore

$$\boxed{E = \frac{\hbar^2 k^2}{2m}}$$

5.2 Particle in an infinite square well

Boundary conditions:

$$V(x) = \begin{cases} 0, & 0 \leq x \leq a, \\ \infty, & \text{elsewhere.} \end{cases}$$

Inside the well:

$$\frac{d^2 \psi}{dx^2} + k^2 \psi = 0$$

General solution:

$$\psi(x) = A \sin(kx) + B \cos(kx)$$

Boundary at $x = 0$:

$$\psi(0) = A \sin(0) + B \cos(0) = B$$

$$\boxed{B = 0}$$

Boundary at $x = a$:

$$\psi(a) = A \sin(ka) = 0$$

For non-zero A :

$$\sin(ka) = 0 \quad \Rightarrow \quad ka = n\pi$$

$$k = \frac{n\pi}{a}$$

Energy:

$$E = \frac{\hbar^2 k^2}{2m}$$

$$E_n = \frac{\hbar^2}{2m} \left(\frac{n\pi}{a} \right)^2$$

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}$$

Equivalent form using $h = 2\pi\hbar$:

$$E_n = \frac{n^2 h^2}{8mL^2}$$

5.3 Energy transition in a box

For a transition:

$$\Delta E = E_{n_f} - E_{n_i}$$

The notes use

$$n_i = 1, \quad n_f = 4, \quad n_f - n_i = 4 - 1 = 3$$

Energy difference:

$$\Delta E = \frac{n_f^2 h^2}{8mL^2} - \frac{n_i^2 h^2}{8mL^2}$$

$$\Delta E = \frac{(n_f^2 - n_i^2) h^2}{8mL^2}$$

Example substitution shown:

$$\Delta E = \frac{(4^2 - 1^2)(6.626070 \times 10^{-34} \text{ J} \cdot \text{s})^2}{8(9.11 \times 10^{-31} \text{ kg})(5 \text{ nm})^2}$$

$$\Delta E = 3.61 \times 10^{-20} \text{ J}$$

5.4 Particle-in-a-box normalization

Wavefunction:

$$\psi_n(x) = A \sin\left(\frac{n\pi x}{L}\right), \quad 0 \leq x \leq L$$

Normalization:

$$\int_0^L |\psi_n(x)|^2 dx = 1$$

$$\int_0^L \left[A \sin\left(\frac{n\pi x}{L}\right) \right]^2 dx = 1$$

$$A^2 \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx = 1$$

Using

$$\sin^2(A) = \frac{1 - \cos(2A)}{2},$$

$$A^2 \int_0^L \frac{1}{2} \left[1 - \cos\left(\frac{2n\pi x}{L}\right) \right] dx = 1$$

The standard result is

$$A^2 \frac{L}{2} = 1$$

$$A = \sqrt{\frac{2}{L}}$$

Thus

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

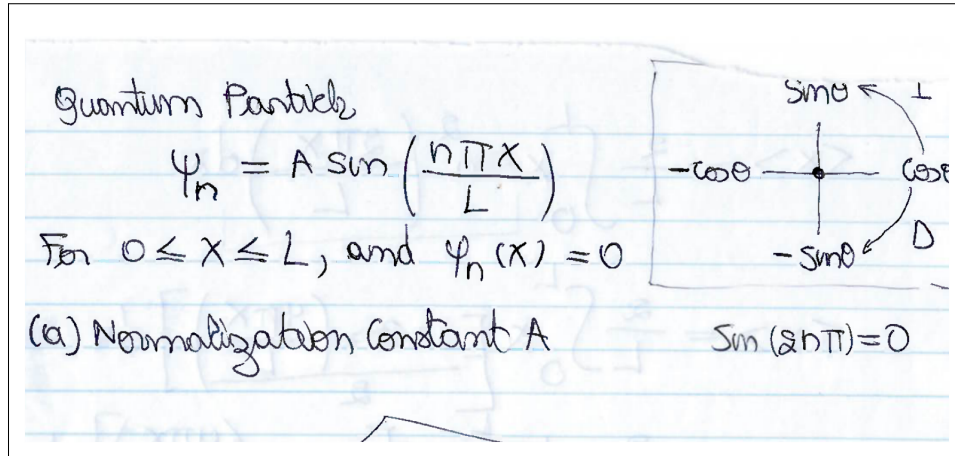


Figure 8: Original particle-in-a-box sketch reused from the handwritten notes.

5.5 Expectation value for the particle in a box

For $n = 2$:

$$\psi_2(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right)$$

Expectation value:

$$\langle x \rangle = \int_0^L x |\psi_2(x)|^2 dx$$

$$\langle x \rangle = \int_0^L x \left[\sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right) \right]^2 dx$$

$$\langle x \rangle = \frac{2}{L} \int_0^L x \sin^2\left(\frac{2\pi x}{L}\right) dx$$

Use

$$\sin^2 A = \frac{1 - \cos(2A)}{2}$$

$$\langle x \rangle = \frac{2}{L} \cdot \frac{1}{2} \int_0^L x \left[1 - \cos\left(\frac{4\pi x}{L}\right) \right] dx$$

$$\langle x \rangle = \frac{1}{L} \left[\int_0^L x dx - \int_0^L x \cos\left(\frac{4\pi x}{L}\right) dx \right]$$

Let

$$k = \frac{4\pi}{L}$$

$$\int x \cos(kx) dx = \frac{x \sin(kx)}{k} + \frac{\cos(kx)}{k^2}$$

Then

$$\left[\frac{x \sin(kx)}{k} + \frac{\cos(kx)}{k^2} \right]_0^L = 0$$

Therefore

$$\langle x \rangle = \frac{1}{L} \left[\frac{L^2}{2} - 0 \right]$$

$$\boxed{\langle x \rangle = \frac{L}{2}}$$

General solution:

$$\psi(x) = A \sin(kx) + B \cos(kx).$$

Boundary conditions:

$$\psi(0) = 0 \Rightarrow B = 0, \quad \psi(L) = 0 \Rightarrow \sin(kL) = 0.$$

Thus

$$kL = n\pi, \quad k = \frac{n\pi}{L}.$$

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2} = \frac{n^2 \hbar^2}{8mL^2}$$

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

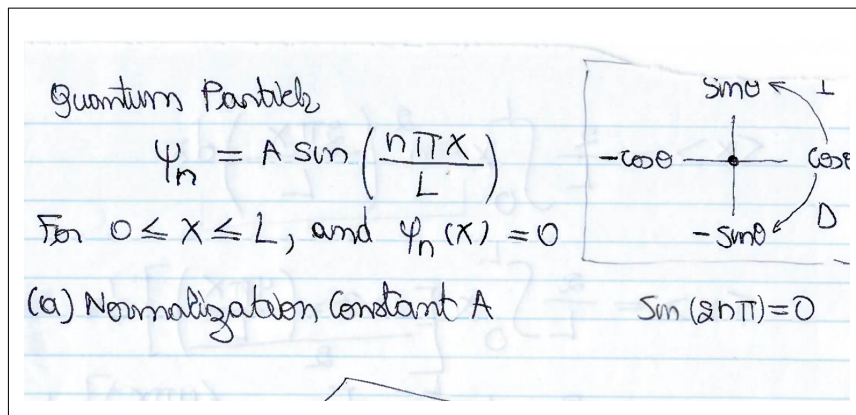


Figure 11: Original particle-in-a-box sketch reused from the concept notes.

11.2 Normalization of ψ_n

Example: normalize $\psi_n = A \sin(n\pi x/L)$

$$\int_0^L |\psi_n(x)|^2 dx = 1.$$

$$A^2 \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx = 1.$$

Use

$$\sin^2 u = \frac{1}{2}(1 - \cos 2u).$$

$$A^2 \int_0^L \frac{1}{2} \left[1 - \cos\left(\frac{2n\pi x}{L}\right) \right] dx = 1.$$

The cosine integral is zero over 0 to L , so

$$A^2 \frac{L}{2} = 1.$$

$$A = \sqrt{\frac{2}{L}}$$

11.3 Expectation value of position

Example: first excited state $n = 2$

$$\psi_2(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right).$$

$$\langle x \rangle = \int_0^L x |\psi_2(x)|^2 dx.$$

$$\langle x \rangle = \frac{2}{L} \int_0^L x \sin^2\left(\frac{2\pi x}{L}\right) dx.$$

Use

$$\sin^2 A = \frac{1}{2}(1 - \cos 2A).$$

$$\langle x \rangle = \frac{1}{L} \int_0^L x \left[1 - \cos\left(\frac{4\pi x}{L}\right)\right] dx.$$

$$\langle x \rangle = \frac{1}{L} \left[\int_0^L x dx - \int_0^L x \cos\left(\frac{4\pi x}{L}\right) dx \right].$$

Let

$$k = \frac{4\pi}{L}.$$

$$\int x \cos(kx) dx = \frac{x \sin(kx)}{k} + \frac{\cos(kx)}{k^2}.$$

Over 0 to L :

$$\int_0^L x \cos\left(\frac{4\pi x}{L}\right) dx = 0.$$

Thus

$$\langle x \rangle = \frac{1}{L} \left[\frac{x^2}{2} \right]_0^L = \frac{1}{L} \frac{L^2}{2}.$$

$$\boxed{\langle x \rangle = \frac{L}{2}}$$

11.4 Energy transition and emitted photon

Example: transition from $n = 4$ to $n = 1$, $L = 5$ nm

$$\Delta E = E_4 - E_1 = \frac{(4^2 - 1^2)h^2}{8mL^2}.$$

$$\Delta E = \frac{15(6.626 \times 10^{-34})^2}{8(9.11 \times 10^{-31})(5 \times 10^{-9})^2}.$$

$$\boxed{\Delta E = 3.61 \times 10^{-20} \text{ J}}$$

Example: wavelength for transition from $n = 5$ to $n = 2$, $L = 5$ nm

$$\Delta E = E_5 - E_2 = \frac{(5^2 - 2^2)h^2}{8mL^2} = \frac{21h^2}{8mL^2}.$$

Photon relation:

$$\Delta E = \frac{hc}{\lambda}.$$

$$\lambda = \frac{hc}{\Delta E} = \frac{8mL^2}{21h}.$$

$$\lambda = \frac{8(9.11 \times 10^{-31})(3.00 \times 10^8)(5 \times 10^{-9})^2}{21(6.626 \times 10^{-34})}.$$

$$\lambda = 3.93 \times 10^{-6} \text{ m} = 3.93 \mu\text{m}$$

11.5 Probability in a region

Example: probability for $0 < x < L/4$ in the first excited state

For $n = 2$:

$$\psi_2(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right).$$

$$P\left(0 < x < \frac{L}{4}\right) = \int_0^{L/4} |\psi_2(x)|^2 dx.$$

$$P = \frac{2}{L} \int_0^{L/4} \sin^2\left(\frac{2\pi x}{L}\right) dx.$$

Use

$$\sin^2 u = \frac{1}{2}(1 - \cos 2u).$$

$$P = \frac{1}{L} \int_0^{L/4} \left[1 - \cos\left(\frac{4\pi x}{L}\right)\right] dx.$$

$$P = \frac{1}{L} \left[x - \frac{L}{4\pi} \sin\left(\frac{4\pi x}{L}\right) \right]_0^{L/4}.$$

$$P = \frac{1}{L} \left(\frac{L}{4} - 0 \right) = \frac{1}{4}.$$

$$P = 0.250 = 25.0\%$$

Example: probability near $L/2$ for ψ_2

Find

$$P(0.490L \leq x \leq 0.510L).$$

Let

$$u = \frac{x}{L}, \quad dx = L du.$$

$$P = 2 \int_{0.490}^{0.510} \sin^2(2\pi u) du.$$

$$P = \left[u - \frac{\sin(4\pi u)}{4\pi} \right]_{0.490}^{0.510}.$$

$$P = 5.26 \times 10^{-5}$$

This is very small because $x = L/2$ is a node for $n = 2$.

Chapter 12 Hydrogen Atom and Angular Momentum

12.1 Hydrogen ground-state radial probability

Concept

For hydrogen ground state:

$$\psi_{100} = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}.$$

Radial probability density:

$$P(r) = 4\pi r^2 |\psi_{100}|^2.$$

Thus

$$P(r) = \frac{4r^2}{a_0^3} e^{-2r/a_0}$$

Example: most probable radius and probability from 0 to a_0

Most probable radius occurs when

$$\frac{dP}{dr} = 0.$$

Since

$$P(r) = \frac{4}{a_0^3} r^2 e^{-2r/a_0},$$

$$\frac{dP}{dr} = \frac{4}{a_0^3} e^{-2r/a_0} \left(2r - \frac{2r^2}{a_0} \right) = 0.$$

$$2r \left(1 - \frac{r}{a_0} \right) = 0.$$

$$r = 0 \quad \text{or} \quad r = a_0.$$

The most probable nonzero radius is

$$r = a_0$$

Probability from 0 to a_0 :

$$P(0 < r < a_0) = \int_0^{a_0} \frac{4r^2}{a_0^3} e^{-2r/a_0} dr.$$

Let

$$x = \frac{2r}{a_0}, \quad r = \frac{a_0 x}{2}, \quad dr = \frac{a_0}{2} dx.$$

Limits:

$$r = 0 \Rightarrow x = 0, \quad r = a_0 \Rightarrow x = 2.$$

$$P = \frac{1}{2} \int_0^2 x^2 e^{-x} dx.$$

Using

$$\int x^2 e^{-x} dx = -e^{-x}(x^2 + 2x + 2),$$

$$P = \frac{1}{2} [-e^{-x}(x^2 + 2x + 2)]_0^2.$$

$$P = \frac{1}{2} [-10e^{-2} + 2] = 1 - 5e^{-2}.$$

$$P = 0.323$$

12.2 Angular momentum for $l = 2$

Concept

Total orbital angular momentum:

$$L = \sqrt{l(l+1)}\hbar.$$

Allowed z components:

$$L_z = m\hbar, \quad m = -l, -l+1, \dots, 0, \dots, l-1, l.$$

Example: $l = 2$

$$L = \sqrt{2(2+1)}\hbar = \sqrt{6}\hbar.$$

$$L = \sqrt{6}\hbar$$

Allowed magnetic quantum numbers:

$$m = -2, -1, 0, 1, 2.$$

Therefore

$$L_z = -2\hbar, -\hbar, 0, \hbar, 2\hbar$$

Final Formula Index

Topic	Formula
Coulomb field	$\mathbf{E} = k_e \frac{Q}{R^3} \mathbf{R}$
Total charge	$Q = \iiint \rho_v dV$
Divergence	$\rho_v = \nabla \cdot \mathbf{D}$
Gauss law	$\oint \mathbf{D} \cdot d\mathbf{S} = Q_{\text{enc}}$
Coaxial electric field	$\mathbf{E} = \frac{a\sigma}{\epsilon_0 \rho} \hat{\mathbf{a}}_\rho$
Coaxial capacitance	$C = \frac{2\pi\epsilon_0 L}{\ln(b/a)}$
Potential	$V(r) = \frac{Q}{4\pi\epsilon_0 r}$
Dipole potential	$V = \frac{k_e p \cos \theta}{r^2}$
Biot-Savart	$d\mathbf{H} = \frac{I d\mathbf{l} \times \mathbf{R}}{4\pi R^3}$
Straight wire	$\mathbf{H} = \frac{I}{2\pi\rho} \hat{\mathbf{a}}_\phi$
Circular loop axis	$\mathbf{H} = \frac{Ia^2}{2(a^2 + z^2)^{3/2}} \hat{\mathbf{a}}_z$
Maxwell wave equation	$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$
Critical angle	$\theta_c = \sin^{-1} \left(\frac{n_2}{n_1} \right)$
Brewster angle	$\tan \theta_B = \frac{n_2}{n_1}$
Double-slit spacing	$\Delta y = \frac{x\lambda}{d}$
Double-slit intensity	$I = I_{\text{max}} \cos^2(\phi/2)$
Compton shift	$\lambda' = \lambda + \frac{h}{m_e c} (1 - \cos \theta)$

10 Quantum Mechanics: Particle in a Box

10.1 Infinite square well

Concept

For

$$0 \leq x \leq L, \quad V(x) = 0,$$

Schrodinger equation becomes

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} = E\psi.$$

Rearrange:

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0, \quad k^2 = \frac{2mE}{\hbar^2}.$$

General solution:

$$\psi(x) = A \sin(kx) + B \cos(kx).$$

Boundary conditions:

$$\psi(0) = 0 \Rightarrow B = 0, \quad \psi(L) = 0 \Rightarrow \sin(kL) = 0.$$

Therefore:

$$kL = n\pi, \quad k = \frac{n\pi}{L}.$$

Energy:

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} = \frac{n^2h^2}{8mL^2}.$$

Normalized wavefunction:

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right).$$

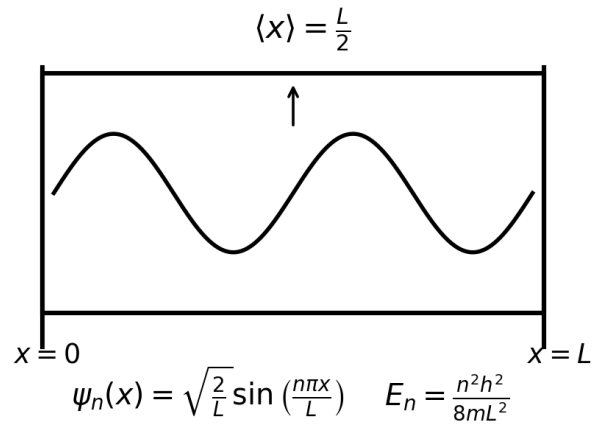


Figure 10.1: Clean particle-in-a-box sketch used for normalization and expectation values.

10.2 Symmetric infinite well about the origin

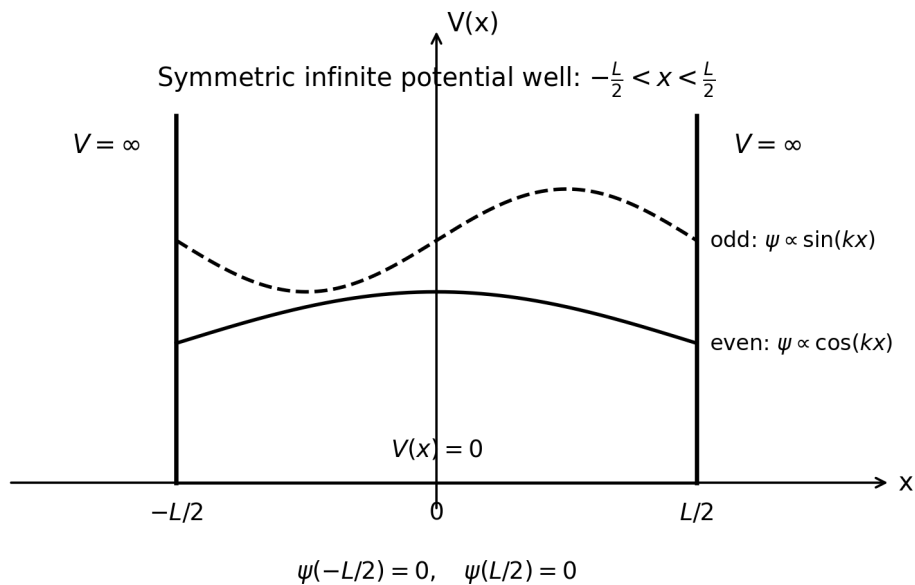


Figure 10.2: Clean symmetric infinite-well diagram showing the boundaries $-L/2$ and $L/2$, with even and odd states.

Example: symmetric infinite potential well

An electron is confined in the one-dimensional infinite potential well

$$V(x) = \begin{cases} 0, & -\frac{L}{2} < x < \frac{L}{2}, \\ \infty, & \text{otherwise.} \end{cases}$$

Find the normalized wavefunctions for both odd and even states and derive the quantized energy levels.

Inside the well

$$V(x) = 0.$$

The time-independent Schrodinger equation becomes

$$-\frac{\hbar^2}{2m_e} \frac{d^2\psi}{dx^2} = E\psi.$$

Rearrange:

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0, \quad k^2 = \frac{2m_e E}{\hbar^2}.$$

General solution:

$$\psi(x) = A \sin(kx) + B \cos(kx).$$

Boundary conditions:

$$\psi\left(-\frac{L}{2}\right) = 0, \quad \psi\left(\frac{L}{2}\right) = 0.$$

Odd states

For odd states:

$$\psi_{\text{odd}}(x) = A \sin(kx).$$

Apply the boundary condition at $x = L/2$:

$$\psi_{\text{odd}}\left(\frac{L}{2}\right) = 0.$$

$$A \sin\left(\frac{kL}{2}\right) = 0.$$

For $A \neq 0$:

$$\sin\left(\frac{kL}{2}\right) = 0.$$

Therefore

$$\frac{kL}{2} = n\pi, \quad k = \frac{2n\pi}{L}, \quad n = 1, 2, 3, \dots$$

So

$$\psi_{\text{odd},n}(x) = A \sin\left(\frac{2n\pi x}{L}\right).$$

Normalize:

$$\int_{-L/2}^{L/2} |\psi_{\text{odd},n}(x)|^2 dx = 1.$$

$$A^2 \int_{-L/2}^{L/2} \sin^2\left(\frac{2n\pi x}{L}\right) dx = 1.$$

Using

$$\int_{-L/2}^{L/2} \sin^2\left(\frac{2n\pi x}{L}\right) dx = \frac{L}{2},$$

we get

$$A^2 \frac{L}{2} = 1, \quad A = \sqrt{\frac{2}{L}}.$$

$$\psi_{\text{odd},n}(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{2n\pi x}{L}\right)$$

Even states

For even states:

$$\psi_{\text{even}}(x) = B \cos(kx).$$

Apply the boundary condition at $x = L/2$:

$$\psi_{\text{even}}\left(\frac{L}{2}\right) = 0.$$

$$B \cos\left(\frac{kL}{2}\right) = 0.$$

For $B \neq 0$:

$$\cos\left(\frac{kL}{2}\right) = 0.$$

Therefore

$$\frac{kL}{2} = \frac{(2n-1)\pi}{2}, \quad k = \frac{(2n-1)\pi}{L}, \quad n = 1, 2, 3, \dots$$

So

$$\psi_{\text{even},n}(x) = B \cos\left(\frac{(2n-1)\pi x}{L}\right).$$

Normalize:

$$\int_{-L/2}^{L/2} |\psi_{\text{even},n}(x)|^2 dx = 1.$$

$$B^2 \int_{-L/2}^{L/2} \cos^2\left(\frac{(2n-1)\pi x}{L}\right) dx = 1.$$

Using

$$\int_{-L/2}^{L/2} \cos^2\left(\frac{(2n-1)\pi x}{L}\right) dx = \frac{L}{2},$$

we get

$$B^2 \frac{L}{2} = 1, \quad B = \sqrt{\frac{2}{L}}.$$

$$\psi_{\text{even},n}(x) = \sqrt{\frac{2}{L}} \cos\left(\frac{(2n-1)\pi x}{L}\right)$$

Energy levels

From

$$k^2 = \frac{2m_e E}{\hbar^2},$$

we get

$$E = \frac{\hbar^2 k^2}{2m_e}.$$

For the combined sequence,

$$k_n = \frac{n\pi}{L}, \quad n = 1, 2, 3, \dots$$

Therefore

$$E_n = \frac{\hbar^2}{2m_e} \left(\frac{n\pi}{L}\right)^2.$$

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2m_e L^2} = \frac{n^2 h^2}{8m_e L^2}$$

Combined final form

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{L}} \cos\left(\frac{n\pi x}{L}\right), & n = 1, 3, 5, \dots, \\ \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), & n = 2, 4, 6, \dots \end{cases}$$

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{L}} \cos\left(\frac{n\pi x}{L}\right), & n = 1, 3, 5, \dots, \\ \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), & n = 2, 4, 6, \dots, \end{cases} \quad E_n = \frac{n^2 \pi^2 \hbar^2}{2m_e L^2}$$

10.3 Normalization of ψ_n

Example: normalize the wavefunction

$$\int_0^L |\psi_n(x)|^2 dx = 1.$$

Substitute:

$$A^2 \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx = 1.$$

Use

$$\sin^2 u = \frac{1}{2}(1 - \cos 2u).$$

Then

$$A^2 \int_0^L \frac{1}{2} \left[1 - \cos\left(\frac{2n\pi x}{L}\right) \right] dx = 1.$$

The cosine integral is zero over 0 to L , so

$$A^2 \frac{L}{2} = 1.$$

$$A = \sqrt{\frac{2}{L}}$$

10.4 Expectation value of position

Example: first excited state

$$\psi_2(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right).$$

Expectation value:

$$\langle x \rangle = \int_0^L x |\psi_2(x)|^2 dx$$

$$= \frac{2}{L} \int_0^L x \sin^2 \left(\frac{2\pi x}{L} \right) dx.$$

Use

$$\sin^2 A = \frac{1}{2}(1 - \cos 2A).$$

Then

$$\begin{aligned} \langle x \rangle &= \frac{1}{L} \int_0^L x \left[1 - \cos \left(\frac{4\pi x}{L} \right) \right] dx \\ &= \frac{1}{L} \left[\int_0^L x dx - \int_0^L x \cos \left(\frac{4\pi x}{L} \right) dx \right]. \end{aligned}$$

Let

$$k = \frac{4\pi}{L}.$$

Use

$$\int x \cos(kx) dx = \frac{x \sin(kx)}{k} + \frac{\cos(kx)}{k^2}.$$

Over 0 to L :

$$\int_0^L x \cos \left(\frac{4\pi x}{L} \right) dx = 0.$$

Therefore:

$$\langle x \rangle = \frac{1}{L} \left[\frac{x^2}{2} \right]_0^L = \frac{1}{L} \frac{L^2}{2}.$$

$$\langle x \rangle = \frac{L}{2}$$

10.5 Energy transition

Example: energy transition in the box

Formula

$$\Delta E = E_4 - E_1 = \frac{(4^2 - 1^2)h^2}{8mL^2}.$$

Substitution

$$\Delta E = \frac{15(6.626 \times 10^{-34})^2}{8(9.11 \times 10^{-31})(5 \times 10^{-9})^2}.$$

$$\Delta E = 3.61 \times 10^{-20} \text{ J}$$

10.6 Expectation value of momentum

Example: momentum expectation value

Momentum operator:

$$\hat{P}_x = -i\hbar \frac{d}{dx}.$$

For $n = 2$:

$$\psi_2(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right).$$

Expectation value:

$$\langle P_x \rangle = \int_0^L \psi_2^*(x) \left(-i\hbar \frac{d}{dx}\right) \psi_2(x) dx.$$

Since ψ_2 is real:

$$\langle P_x \rangle = -i\hbar \int_0^L \psi_2(x) \frac{d\psi_2(x)}{dx} dx.$$

Compute derivative:

$$\frac{d\psi_2}{dx} = \sqrt{\frac{2}{L}} \frac{2\pi}{L} \cos\left(\frac{2\pi x}{L}\right).$$

Thus

$$\langle P_x \rangle = -i\hbar \left(\frac{2}{L}\right) \left(\frac{2\pi}{L}\right) \int_0^L \sin\left(\frac{2\pi x}{L}\right) \cos\left(\frac{2\pi x}{L}\right) dx.$$

Use

$$\sin A \cos A = \frac{1}{2} \sin(2A).$$

Then

$$\begin{aligned} \int_0^L \sin\left(\frac{2\pi x}{L}\right) \cos\left(\frac{2\pi x}{L}\right) dx &= \frac{1}{2} \int_0^L \sin\left(\frac{4\pi x}{L}\right) dx \\ &= 0. \end{aligned}$$

$$\boxed{\langle P_x \rangle = 0}$$

3 Charge Density, Divergence, and Total Charge

3.1 Divergence method for enclosed charge

Concept

Use Gauss's law in differential form:

$$\rho_v = \nabla \cdot \mathbf{D}.$$

For a very small volume ΔV :

$$Q_{\text{enc}} \approx (\nabla \cdot \mathbf{D})\Delta V.$$

In Cartesian coordinates:

$$\nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}.$$

Example: incremental charge from flux density

Given

$$\mathbf{D} = e^{-x} \sin y \hat{\mathbf{a}}_x - e^{-x} \cos y \hat{\mathbf{a}}_y + 2z \hat{\mathbf{a}}_z.$$

Find the approximate charge enclosed in $\Delta V = 10^{-3} \text{ m}^3$ at the origin.

Formula

$$Q_{\text{enc}} = (\nabla \cdot \mathbf{D})\Delta V.$$

Divergence

$$\begin{aligned}\nabla \cdot \mathbf{D} &= \frac{\partial}{\partial x}(e^{-x} \sin y) + \frac{\partial}{\partial y}(-e^{-x} \cos y) + \frac{\partial}{\partial z}(2z) \\ &= -e^{-x} \sin y + e^{-x} \sin y + 2 \\ &= 2.\end{aligned}$$

Substitution

$$Q_{\text{enc}} = (2)(10^{-3}) = 2 \times 10^{-3} \text{ C}.$$

$$Q_{\text{enc}} = 2.00 \times 10^{-3} \text{ C}$$

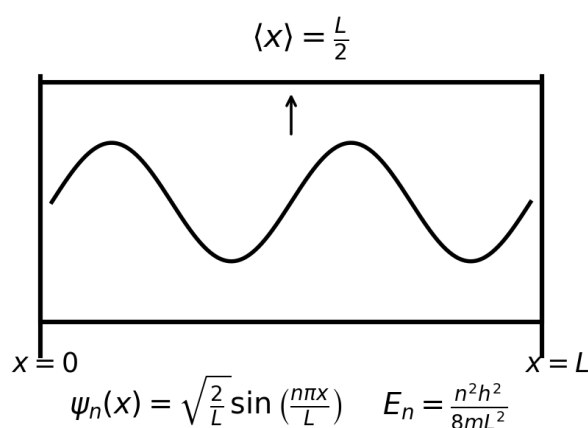


Figure 10.1: Clean particle-in-a-box sketch used for normalization and expectation values.

10.2 Normalization of ψ_n

Example: normalize the wavefunction

$$\int_0^L |\psi_n(x)|^2 dx = 1.$$

Substitute:

$$A^2 \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx = 1.$$

Use

$$\sin^2 u = \frac{1}{2}(1 - \cos 2u).$$

Then

$$A^2 \int_0^L \frac{1}{2} \left[1 - \cos\left(\frac{2n\pi x}{L}\right) \right] dx = 1.$$

The cosine integral is zero over 0 to L , so

$$A^2 \frac{L}{2} = 1.$$

$$A = \sqrt{\frac{2}{L}}$$

10.3 Expectation value of position

Example: first excited state

$$\psi_2(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right).$$

Expectation value:

$$\begin{aligned}\langle x \rangle &= \int_0^L x |\psi_2(x)|^2 dx \\ &= \frac{2}{L} \int_0^L x \sin^2 \left(\frac{2\pi x}{L} \right) dx.\end{aligned}$$

Use

$$\sin^2 A = \frac{1}{2}(1 - \cos 2A).$$

Then

$$\begin{aligned}\langle x \rangle &= \frac{1}{L} \int_0^L x \left[1 - \cos \left(\frac{4\pi x}{L} \right) \right] dx \\ &= \frac{1}{L} \left[\int_0^L x dx - \int_0^L x \cos \left(\frac{4\pi x}{L} \right) dx \right].\end{aligned}$$

Let

$$k = \frac{4\pi}{L}.$$

Use

$$\int x \cos(kx) dx = \frac{x \sin(kx)}{k} + \frac{\cos(kx)}{k^2}.$$

Over 0 to L :

$$\int_0^L x \cos \left(\frac{4\pi x}{L} \right) dx = 0.$$

Therefore:

$$\langle x \rangle = \frac{1}{L} \left[\frac{x^2}{2} \right]_0^L = \frac{1}{L} \frac{L^2}{2}.$$

$$\boxed{\langle x \rangle = \frac{L}{2}}$$

10.4 Energy transition

Example: energy transition in the box

Formula

$$\Delta E = E_4 - E_1 = \frac{(4^2 - 1^2)h^2}{8mL^2}.$$

Substitution

$$\Delta E = \frac{15(6.626 \times 10^{-34})^2}{8(9.11 \times 10^{-31})(5 \times 10^{-9})^2}.$$

$$\boxed{\Delta E = 3.61 \times 10^{-20} \text{ J}}$$

Chapter 7 Compact Formula Index

Source pages: All uploaded note sets; duplicates combined

Category	Formula / Result
Differentiation	$\frac{d}{dx}(x^n) = nx^{n-1}$
Derivative of reciprocal	$\frac{d}{du}(u^{-1}) = -u^{-2}$
Integration by parts	$\int u dv = uv - \int v du$
Gaussian integral	$\int_{-\infty}^{\infty} e^{-bx^2} dx = \sqrt{\pi/b}$
Trig identity	$\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$
Trig identity	$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$
Coulomb force	$\mathbf{F} = k_e \frac{q_1 q_2}{R^3} \mathbf{R}$
Point potential	$V(r) = \frac{q}{4\pi\epsilon_0 r}$
Field from potential	$\mathbf{E} = -\nabla V$
Divergence charge density	$\rho_v = \nabla \cdot \mathbf{D}$
Cylindrical capacitor capacitance	$C = \frac{2\pi\epsilon_0 L}{\ln(b/a)}$
Cylindrical capacitor energy	$W_E = \frac{\pi a^2 \sigma^2 L}{\epsilon_0} \ln\left(\frac{b}{a}\right)$
Biot-Savart law	$d\mathbf{H} = \frac{I d\mathbf{l} \times \mathbf{R}}{4\pi R^3}$
Infinite straight wire	$\mathbf{H} = \frac{I}{2\pi\rho} \hat{\mathbf{a}}_\phi$
Schrodinger equation	$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi = i\hbar \frac{\partial \psi}{\partial t}$
Particle in box energy	$E_n = \frac{n^2 \hbar^2}{8mL^2}$
Particle in box wavefunction	$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$
Expectation value	$\langle x \rangle = \int_0^L x \psi_n(x) ^2 dx$
Double slit bright fringes	$y_m = \frac{xm\lambda}{d}$
Fringe spacing	$\Delta y = \frac{x\lambda}{d}$
Compton shift	$\Delta\lambda = \frac{h}{m_e c} (1 - \cos \theta)$
Photon energy	$E = \frac{hc}{\lambda}$

Stefan-Boltzmann law

$$P = \sigma AT^4$$

Wien's law

$$\lambda_{\max}T = 2.898 \times 10^{-3} \text{ m} \cdot \text{K}$$

FORMULA REFERENCE - CONTINUED

Black body	$P = \sigma AT^4$
Wien law	$\lambda_{\max} T = 2.898 \times 10^{-3} \text{ m} \cdot \text{K}$
Photoelectric effect	$K_{\max} = \frac{hc}{\lambda} - \phi$
Particle in a box energy	$E_n = \frac{n^2 h^2}{8mL^2}$
Particle in a box wavefunction	$\psi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$
Expectation value	$\langle x \rangle = \int x \psi(x) ^2 dx$
Hydrogen radial probability	$P(r) = \frac{4r^2}{a_0^3} e^{-2r/a_0}$
Orbital angular momentum	$L = \sqrt{l(l+1)}\hbar, L_z = m\hbar$

Source Selection Notes

The examples were selected from the uploaded Physics 5 assessments, tests, tutorials, formula sheets, and the previously prepared concept documentation. Questions outside the already documented topics were not expanded into full chapters.

11 Drawing Checklist

Question type	Drawing required
Coulomb force / electric field superposition	3D coordinate sketch showing charges, observation point, and relative vectors $\mathbf{R}_i$.
Cylindrical beam / total charge	Cylinder with radius/diameter, z -limits, and cylindrical coordinates (ρ, ϕ, z) .
Cylindrical dielectric / coaxial capacitor	Gaussian cylinder, inner/outer radii, and field direction $\hat{\mathbf{a}}_\rho$.
Annular sheet	Ring/disk in the xy -plane and point $P(0, 0, z)$ on the axis.
Point-charge potential	Radial path from infinity to r , with $d\mathbf{l} = dr\hat{\mathbf{a}}_r$.
Biot-Savart straight wire	Wire along z -axis, current direction, observation point, and vector $\mathbf{R}$.
Critical/Brewster angle	Interface, normal, incident, reflected, and refracted rays.
Double-slit	Slit separation d , screen distance x , fringe position y , and angle θ .
Particle in a box	Box from $x = 0$ to $x = L$, wave pattern, and symmetry point $L/2$.
Symmetric infinite well	Box from $x = -L/2$ to $x = L/2$, origin at the center, even/odd wave shapes, and boundary conditions.

12 Final Formula Index

Topic	Formula
Coulomb electric field	$\mathbf{E} = k_e \frac{Q}{R^3} \mathbf{R}$
Coulomb force	$\mathbf{F} = k_e \frac{q_1 q_2}{R^3} \mathbf{R}$
Total charge	$Q = \iiint_V \rho_v \, dV$
Cylindrical volume element	$dV = \rho \, d\rho \, d\phi \, dz$
Divergence	$\rho_v = \nabla \cdot \mathbf{D}$
Gauss law	$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_{\text{enc}}$
Coaxial electric field	$\mathbf{E} = \frac{\sigma a}{\varepsilon_0 \rho} \hat{\mathbf{a}}_\rho$
Coaxial capacitance	$C = \frac{2\pi\varepsilon_0 L}{\ln(b/a)}$
Coaxial energy	$W_E = \frac{\pi a^2 \sigma^2 L}{2} \ln(b/a)$
Annular sheet field	$\mathbf{E} = \frac{\sigma z}{2\varepsilon_0} \left[\frac{1}{\sqrt{a^2 + z^2}} - \frac{1}{\sqrt{b^2 + z^2}} \right] \hat{\mathbf{a}}_z$
Point-charge potential	$V(r) = \frac{Q}{4\pi\varepsilon_0 r}$
Biot-Savart law	$d\mathbf{H} = \frac{I \, d\mathbf{l} \times \mathbf{R}}{4\pi R^3}$
Straight wire field	$\mathbf{H} = \frac{I}{2\pi\rho} \hat{\mathbf{a}}_\phi$
Critical angle	$\theta_c = \sin^{-1} \left(\frac{n_2}{n_1} \right)$
Brewster angle	$\tan \theta_B = \frac{n_2}{n_1}$
Double-slit spacing	$\Delta y = \frac{x\lambda}{d}$
Double-slit intensity	$I = I_{\text{max}} \cos^2(\phi/2)$
Phase-path relation	$\phi = \frac{2\pi}{\lambda} \Delta$
Black-body radiation	$P = \sigma AT^4$
Wien law	$\lambda_{\text{max}} T = 2.898 \times 10^{-3} \, \text{m K}$
Photoelectric effect	$K_{\text{max}} = \frac{hc}{\lambda} - \phi$
Compton shift	$\lambda' = \lambda + \frac{h}{m_e c} (1 - \cos \theta)$
Particle-in-box energy	$E_n = \frac{n^2 h^2}{8mL^2}$
Particle-in-box wavefunction	$\psi_n(x) = \sqrt{\frac{2}{L}} \sin \left(\frac{n\pi x}{L} \right)$

FINAL FORMULA INDEX - CONTINUED

Symmetric well even states	$\psi_{\text{even},n} = \sqrt{\frac{2}{L}} \cos\left(\frac{(2n-1)\pi x}{L}\right)$
Symmetric well odd states	$\psi_{\text{odd},n} = \sqrt{\frac{2}{L}} \sin\left(\frac{2n\pi x}{L}\right)$
Expectation value	$\langle x \rangle = \int x \psi(x) ^2 dx$
Momentum operator	$\hat{P}_x = -i\hbar \frac{d}{dx}$

REFERENCES

HNK / MindBuilders | South Africa | Websites and contact links

[1] HNK IoT Solutions - official website

<https://hnkiot.com/>

[2] LinkedIn article

<https://lnkd.in/p/dWhpzZPa>

[3] Learn Vectors

<https://learn-vectors.netlify.app/>

[4] Learn Maths Engineering 5

<https://learn-mathseng5.netlify.app/>

[5] AI Learning Hub

<https://learn-ailearninghub.netlify.app/>

[6] UI Builders

<https://uibuilders.netlify.app/>

[7] Canvas / WebGL / Three.js

<http://canvas-webglthreejs.netlify.app/>

[8] WhatsApp - +27 61 471 9400

<https://wa.me/27614719400>

[9] Email - hello@hnkiot.com

<mailto:hello@hnkiot.com>

[10] Engineering portfolio - hnk-engineering-portfolio.netlify.app

<https://hnk-engineering-portfolio.netlify.app/>

[11] YouTube playlists - @HNK2005 / playlists

<https://www.youtube.com/@HNK2005/playlists>

[12] TikTok - @codingfever

https://www.tiktok.com/@codingfever?is_from_webapp=1&sender_device=pc

REFERENCES - CONTINUED

HNK / MindBuilders | South Africa | Websites and contact links

[13] Instagram - @hhnk.3693

<https://www.instagram.com/hhnk.3693/>

[14] TikTok - @horizonhnk

<https://www.tiktok.com/@horizonhnk>

[15] X - @HnkHorizon

<https://x.com/HnkHorizon>

[16] GitHub - henockhnk092-dot

<https://github.com/henockhnk092-dot>

[17] GitHub - hnkiot

<https://github.com/hnkiot>

[18] LinkedIn - HNK IoT Solutions

<https://www.linkedin.com/in/hnk-iot-solutions/?isSelfProfile=false>

[19] Instagram - @hnkiotsolutions

<https://www.instagram.com/hnkiotsolutions/>

[20] Facebook - HnkIoT Solutions

<https://www.facebook.com/HnkIoT Solutions>

[21] YouTube - @HNKIOT Solutions

<https://www.youtube.com/@HNKIOT Solutions>

[22] TikTok - @hnkiot

<https://www.tiktok.com/@hnkiot>

[23] Pinterest - hnkiotsolutions

<https://za.pinterest.com/hnkiotsolutions/>

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B E C O M E R E A D Y

Summary: This Physics 5 roadmap covers Vector Calculus and Dynamic Systems, bridging theory and practical engineering for students and professionals.

Subtitle: Master the Art of Vector Calculus. Learn, Analyze, and Explore the Mathematics of Dynamic Systems.

Author information: HNK, the founder of Fix Master, is a Computer Engineer dedicated to innovation in network, electronic, and embedded systems. Based in Cape Town, he leverages a strong foundation in mathematics and scientific computing to solve real-world problems.

Discover how to bring signal graphics to life through dynamic visual effects and smooth calculations.

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